LECTURER NOTES

STRUCTURAL ANALYSIS

B.Tech,4TH Semester,
Civil Engineering

Prepared by:
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COURSE CONTENT

Structural Analysis

B.Tech, 4th Semester, CE

Module- I

Concept of determinate and indeterminate structures, determination of degree of static andkinematic indeterminacy in plane frame and continuous structures.

Methods of Analysis: Equilibrium equations, compatibility requirements, Introduction to force anddisplacement methods.

Analysis of propped cantilever by consistent deformation method, Analysis of fixed andcontinuous beams by Moment-Area method, Conjugate beam method and theorem of threemoments.

Module- II

Energy theorems and its application, Strain energy method, Virtual work method, unit loadmethod, Betti's and Maxwell's laws, Castigliano's theorem, concept of minimum potential energy.Theories of failure, Maximum normal stress theory, maximum normal strain theory, maximum shearing strain theory,maximum strain energy theory, maximum distortion energy theory, maximum octahedral shearingstress theory.

REFERENCES

Structural Analysis

B.Tech, 4th Semester, CE

Books:

[1] C. S. Reddy. Basic structural analysis. McGraw Hill Education. S.S. Bhavikatti, Structural Analysis.Vikas Publishing House

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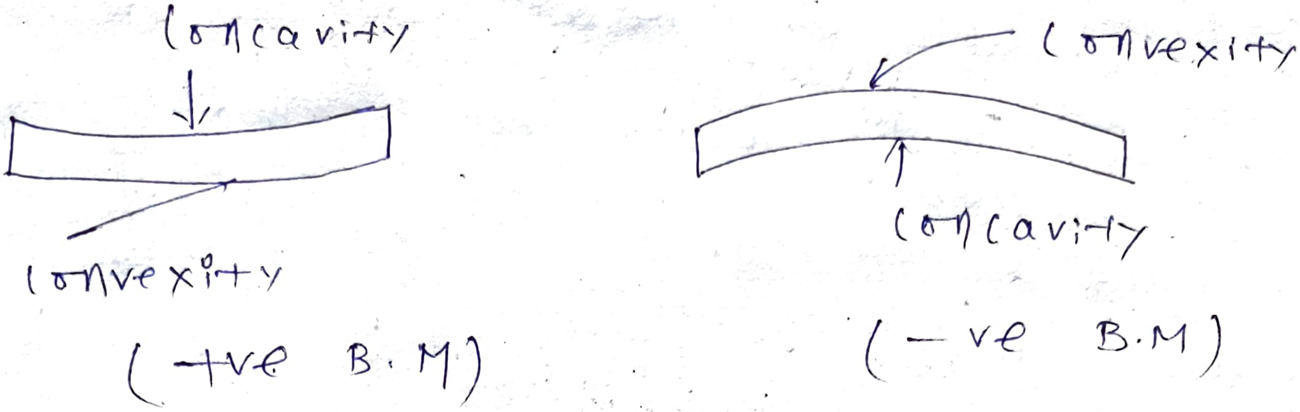


Shear Force :→ (S.F)

The algebraic sum of the vertical forces at any section of a beam to the right or left of the section is known as shear force.

Bending Moment :→ (B.M) $\boxed{M = \text{Force} \times \perp\text{r distance}}$

The algebraic sum of the moments of all the forces acting to the right or left of the section is known as bending moment.

Shear Force Diagram (SFD) :—

It is one which shows the variation of the S.F along the length of the beam.

Bending Moment Diagram (BMD) :—

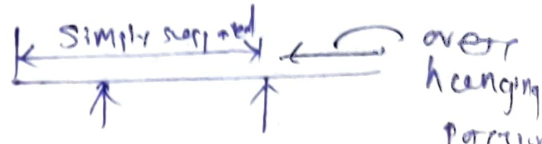
It is one which shows the variation of the bending moment along the length of the beam.

Types of Beams:—

1. Cantilever beam → A beam which is fixed at one end and free at the other end

2. Simply supported Beam → A beam supported freely on the supports at its both ends.

3. Overhanging beam (If the end portion of a beam is extended beyond the support)



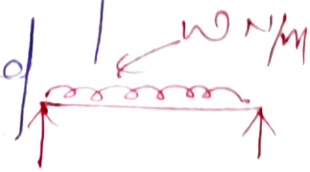
4. Fixed Beam (Both end fixed)

5. Continuous beam (A beam which is provided more than two supports)

Types of load →

1. Concentrated or point load ← It is one which is considered to act at a point, although in practice it must really be distributed over a small area.

2. Uniformly distributed load

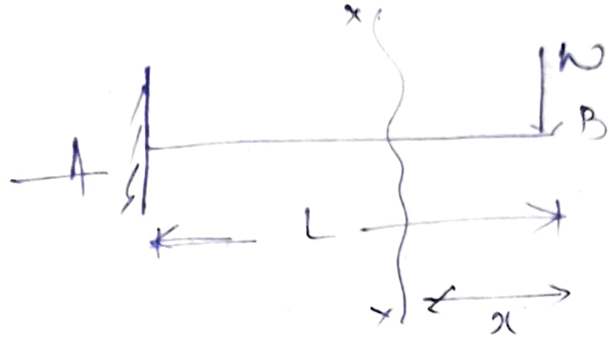


It is one which is spread over a beam in such a manner that rate of loading w is uniform along the length.

3. Uniformly varying load

A uniformly varying load is one which is spread over a beam in which is spread over a beam in such a manner that rate of loading varies from pt. to pt. along the beam in which load is zero at one end and increases uniformly to the other end. Such load is known as triangular load.

Cantilever with a pt. load at the free end :→



S.F at x, $F_x = +W$

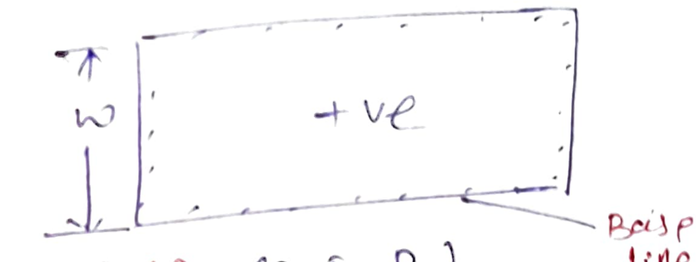
S.F at B, $F_B = +W$

S.F at A, $F_A = +W$

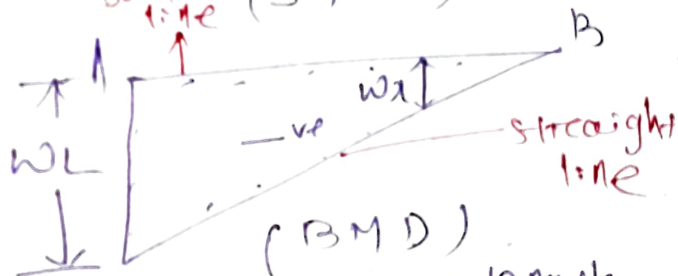
B.M at x, $M_x = -W \times x$

at $x = 0$, $M_B = 0$ (at B)

at $x = L$ at A, $M_A = -WL$

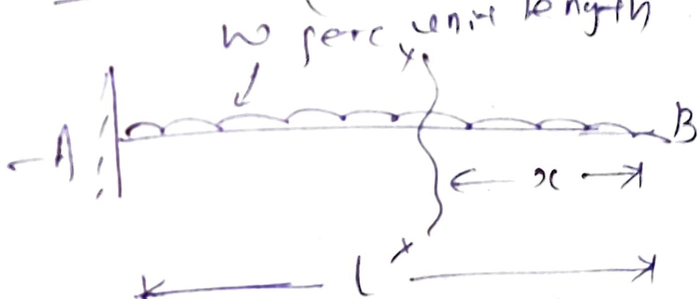


(S.F.D)



(BMD)

Cantilever with a udl



S.F at x, $F_x = w.x$

at $x = 0$, at B, $F_B = 0$

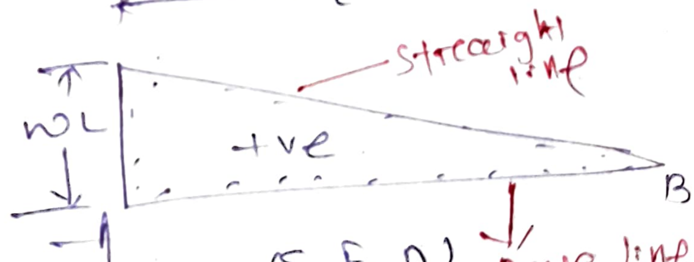
at $x = L$, at A, $F_A = wL$

B.M at x, $M_x = -w.x.\frac{x}{2}$

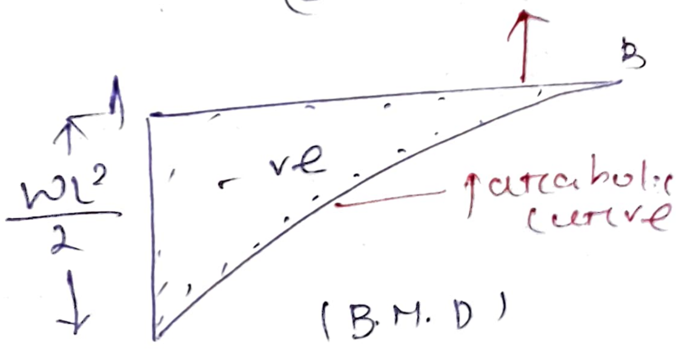
$= -\frac{wx^2}{2}$

at $x = 0$, at B, $M_B = 0$

at $x = L$, at A, $M_A = -\frac{wL^2}{2}$

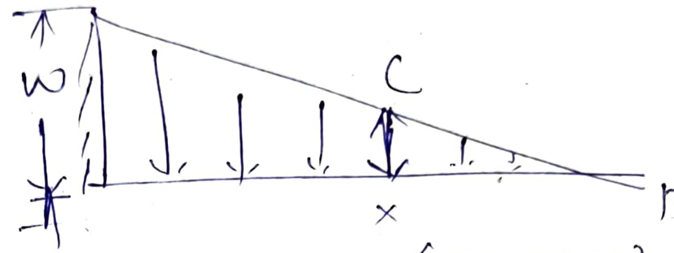


(S.F.D)



(B.M.D)

Cantilever bearing carrying a gradually varying load :—



(Load diagram)

$XB = x$

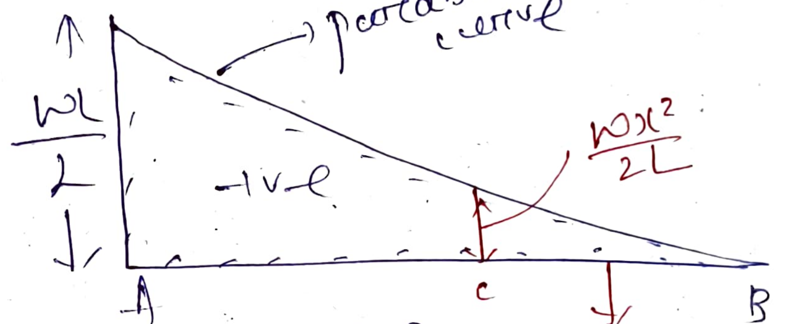
$XC = \frac{wx}{L}$

$$\boxed{F_x = \frac{wx^2}{2L}}$$

F_x = Area of $\triangle BCX$

$= \frac{1}{2} \times XB \times XC$

$= \frac{1}{2} \cdot x \cdot \frac{wx}{L} = \frac{wx^2}{2L}$



(S.F.D)

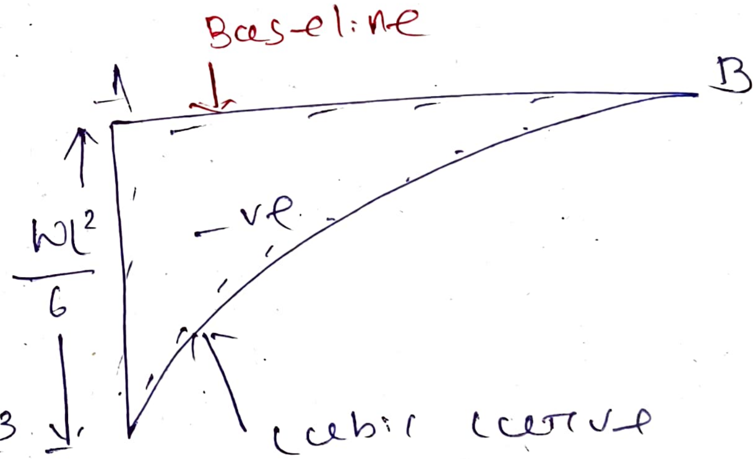
at $x = 0$, at B, $F_B = 0$

at $x = L$, at A, $F_A = \frac{wL^2}{2L}$

$= \frac{wL}{2}$

$M_x = -F_x \times \frac{x}{3}$

$= -\frac{wx^2}{2L} \times \frac{x}{3} = \frac{-wx^3}{6L}$



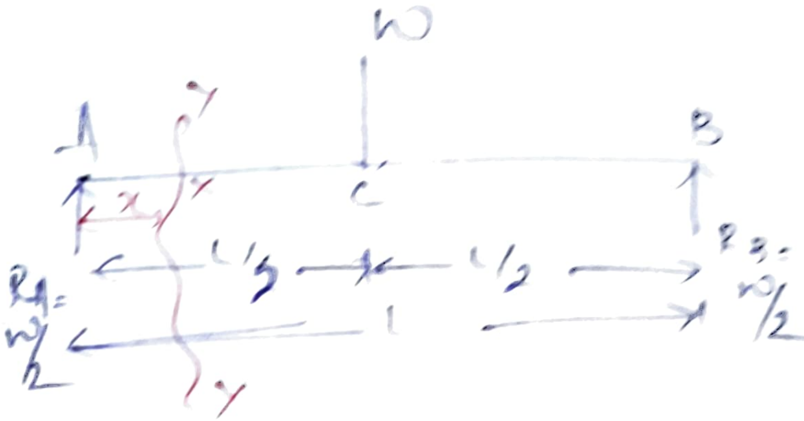
(B.M.D)

at $x = 0$, at B, $M_B = 0$

at $x = L$, ~~at~~ A, $M_A = \frac{-wL^3}{6L}$

$= \frac{-wL^2}{6}$

Simply supported beam with a point load at mid point :—

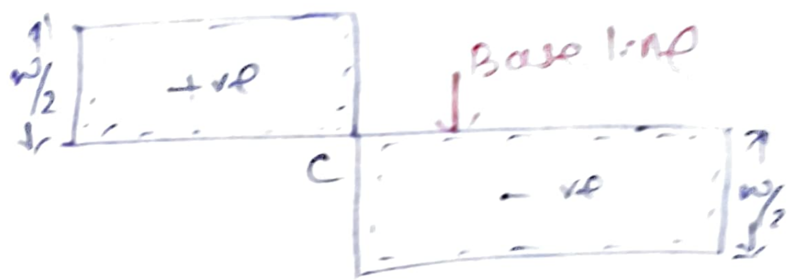


$$R_A = R_B = \frac{W}{2}$$

$$F_x = \frac{W}{2}$$

A to C $\longrightarrow \frac{W}{2}$

C to B $\longrightarrow \frac{W}{2} - W = -\frac{W}{2}$



(S.F.D)

*

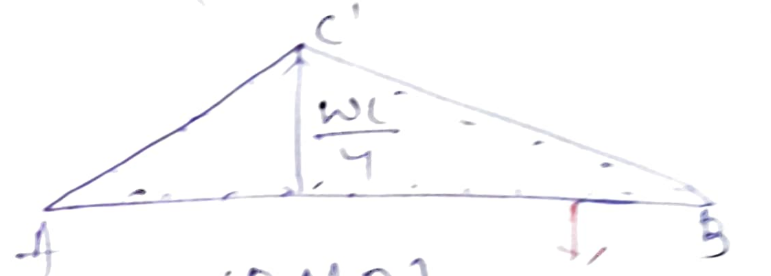
$$M_x = R_A \cdot x$$

$$= \frac{W}{2} \cdot x$$

at $x = 0$, at A, $M_A = \frac{W}{2} \times 0 = 0$

at $x = \frac{L}{2}$, at C, $M_C = \frac{W}{2} \times \frac{L}{2}$

$$= \frac{WL}{4}$$



(B.M.D)

* $M_x = R_A x - W \times (x - \frac{L}{2})$

$$= \frac{W}{2} x - Wx + \frac{WL}{2}$$

$$= \frac{WL}{2} - \frac{2x}{2}$$

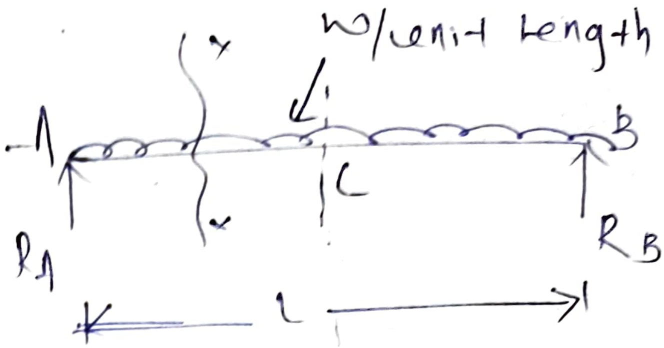
at $x = \frac{L}{2}$, $M_C = \frac{WL}{2} - \frac{WL}{2 \cdot 2}$

$$= \frac{WL}{4}$$

at $x = L$, $M_B = \frac{WL}{2} - \frac{W}{2} \times L$

$$= 0$$

Simply supported beam carrying a udl :—



$R_A = R_B = \frac{wL}{2}$

$F_x = R_A - w.x$

$= \frac{wL}{2} - wx$

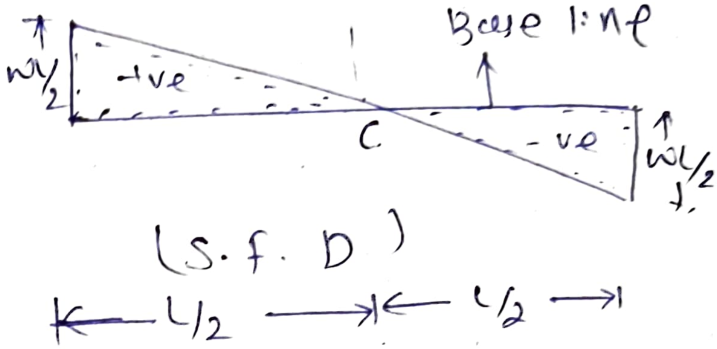
at $x = 0$, at A, $F_A = \frac{wL}{2} - 0 = \frac{wL}{2}$

at B, at $x = L$, $F_B = \frac{wL}{2} - wL$

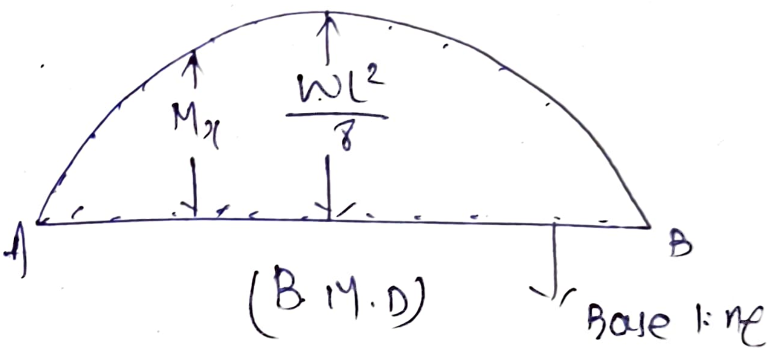
$= -\frac{wL}{2}$

at $x = \frac{L}{2}$, at C, $F_C = \frac{wL}{2} - \frac{wL}{2}$

$= 0$



(S.F.D)



(B.M.D)

$M_x = R_A . x - w.x.\frac{x}{2}$

$= \frac{wL}{2} . x - \frac{wx^2}{2}$

at $x = 0$, at A, $M_A = 0$

at B, $x = L$, $M_B = \frac{wL}{2} . L - \frac{w}{2} . L^2 = 0$

at $x = \frac{L}{2}$, $M_C = \frac{wL}{2} . \frac{L}{2} - \frac{w}{2} \left(\frac{L}{2}\right)^2$

$= \frac{wL^2}{4} - \frac{wL^2}{8}$

$= \frac{wL^2}{8}$

Engineering Mechanics:—

It is a branch of engineering that uses physics and mathematics to analyze and design mechanical systems.

* While transferring the loads acting on the structure, the members of the structure are subjected to internal forces like axial forces, shearing forces, bending and torsional moments.

Structural Analysis:—

It deals with analysing these internal forces in the members of the structures.

Strength of Materials:—

The behaviour of the materials of the structures subjected to different types of internal forces is ~~covered~~, ~~under~~ ~~strt. is ca~~ called strength of materials.

Structural Design:—

It deals with sizing various members of the structure to resist the internal forces to which they are subjected to in the course of their life cycle.

* The analysis of pin-jointed determinate plane frames are in Engineering Mechanics and the determination of bending moment and shear force in determinate beams in Strength of Material.

Idealisations and Assumptions :-

Material properties :-

Homogeneous

It refers to the identical particles that exist throughout the material.

Isotropic

It refers to the physical properties of the material which are identical in all the directions.

* Another Assumption is stress-strain relation is linear, which means in case of metals, the analysis is carried out within the limit of proportionality and in case of materials like concrete, the stress-strain relation is approximated to a linear relation.

Boundary conditions :-

i) Free end :-

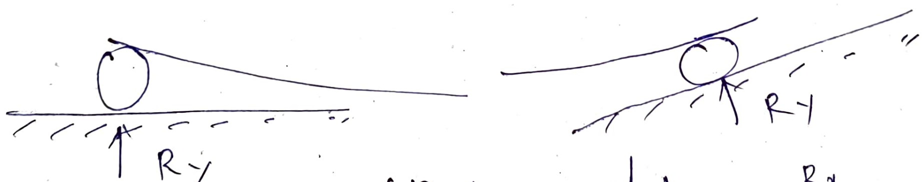
At the free end a structure can have linear or rotational displacement in any direction and hence, no reaction is developed.

Ex:- Free end of a cantilever beam.

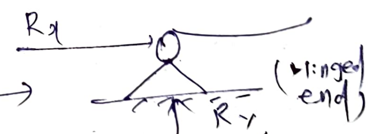
ii) Roller end :-

At such end the member is free to move along the support and can rotate freely.

→ There is no reaction along the support and the resisting moment is zero.



(Roller end)



iii) Hinged ends :-

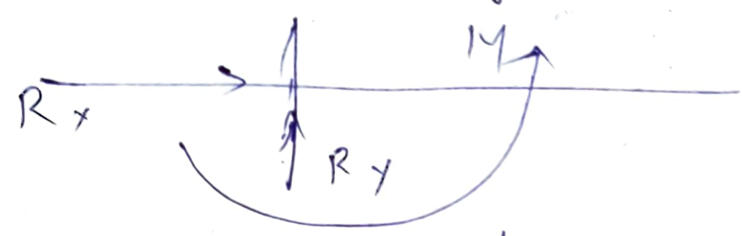
At such ends it is assumed that the member cannot have linear motion in any direction but can rotate freely along the support point, i.e., the end is pinned.

- In this case, the support can develop the required resisting force in any direction and not the resisting moment.

iv) Fixed ends :—

Such ends cannot have any linear or rotational movement.

- At the fixed end, the support can develop not only the resisting force in any direction but also the resisting moment.



(Fixed end)

Small Deflections :—

Deflections are assumed to be small, i.e. the changes in the shape of the structure due to loading are negligible.

- For all calculations, the changes in length of a member and the angle between any two members are neglected.

Loads :—

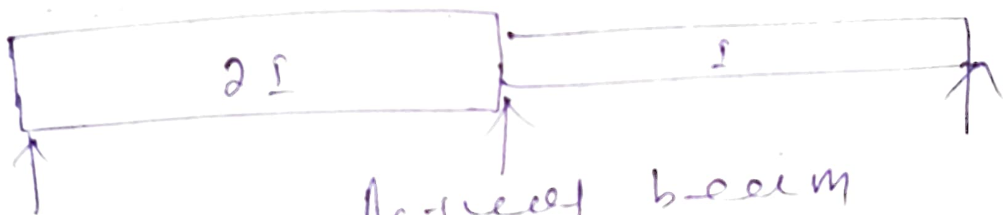
Concentrated load
A heavy load distributed over a small area is assumed as a concentrated load acting at a point.

Uniformly distributed load
Live load includes the weight of persons and all other moveable material (like furnitures), which vary from time to time on the structure.

Idealising the structure :-

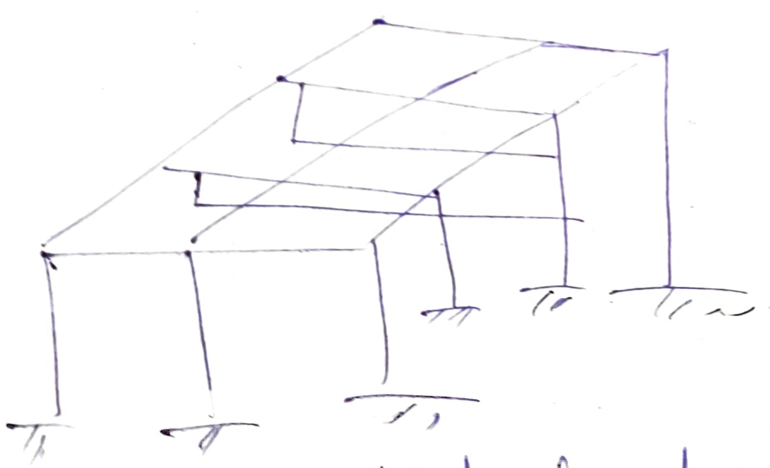
Structures are usually idealised as three-dimensional.
But without losing significant accuracy many structures are idealised as one or two-dimensional which helps in simplifying the analysis considerably.

Ex- A beam idealised as a one-dimensional structure since it has considerable dimension in one direction compared to the dimensions in the other two (cross-sectional) directions.

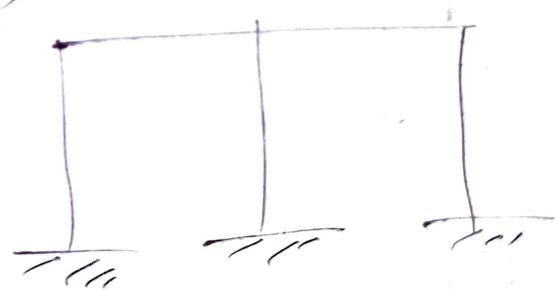


Actual beam

(Idealized beam)



(Idealised space frame)



Idealised plane frame

Law of superposition:-

The law of superposition holds good when the material is assumed to be perfectly elastic and obeys Hooke's law for the range of loads considered.

Conditions of Equilibrium:-

i) the summation of all the forces along any axis is zero.

ii) the summation of all the moments about any axis is also zero.

$$\left.\begin{aligned} \Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma F_z = 0 \\ \Sigma M_x = 0, \quad \Sigma M_y = 0, \quad \Sigma M_z = 0 \end{aligned}\right\} \rightarrow 3D$$

$$\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M = 0 \rightarrow 2D$$

Compatibility conditions:-

1. the members meeting at a joint will continue to meet at the same joint even after deformation takes place.

2. At rigid joints, the angle between any two members remains the same even after deformation takes place.

- the compatibility conditions will help in formulating additional equations.

Statically determinate structures:—

A structure of system which can be analysed by using eqns of statical eqm only is called statically determinate structure.

Ex:— Beams or trusses with both ends simply supported, one end hinged and another on rollers and the cantilever type.

Statically indeterminate structures:— (Redundant structures)

A structure which can't be analysed by using eqns of eqm only is called statically indeterminate structure.

For Ex:— Fixed Beams
continuous Beam
propped cantilevers.

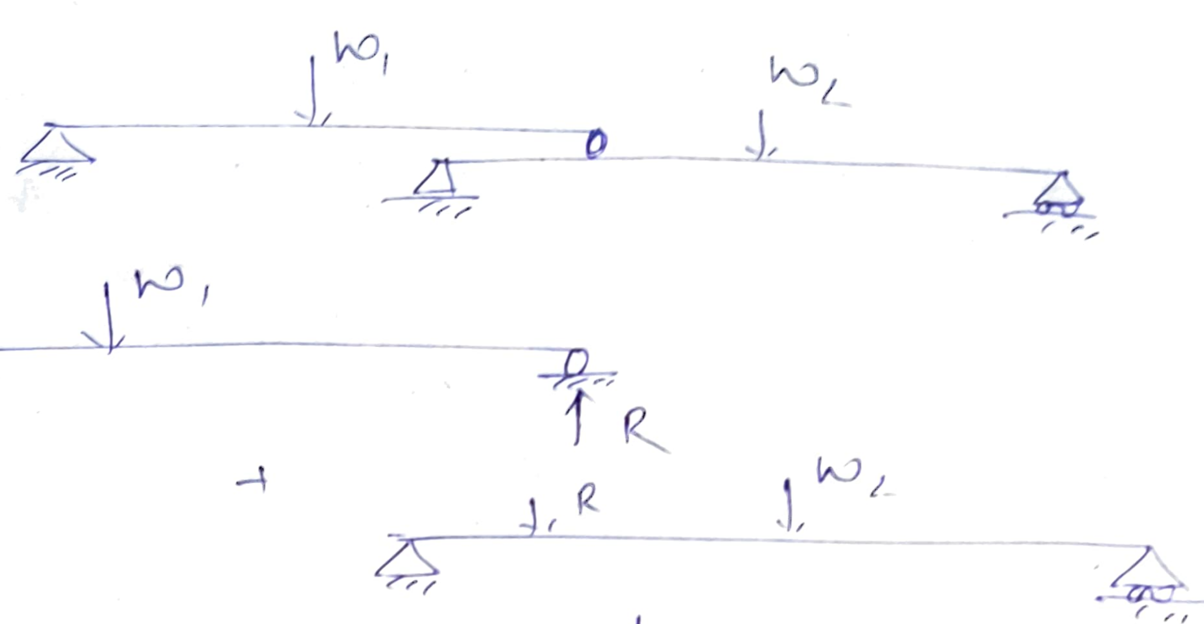
Simple structure:—

A determinate str. is called simple str., if it made of a single system.

compound structure:—

If a system is to be split into a no. of system which may then be analysed with eqns of eqm alone, the system is called compound system.

Ex:-



(compound system)

It may be analysed by using eqn's of eqm only after splitting it into two simple beams.

Linear systems:-

A system is called a linear system if its material has linear stress-strain relationship and a small deformation.

Non-linear systems:-

A system will be treated as a non-linear system if its material doesn't have linear stress-strain relationship or its deformation is so large that a change of geometry can't be neglected in the analysis.

To find the deflections of determinate beams.

1. Double Integration / Macaulay's Method
2. Moment Area Method
3. Conjugate Beam "
4. Strain Energy "
5. Castigliano's "
6. Unit load "

Moment Area theorems :—

Thm-1

The change in the slope between two points of a straight member under flexure is equal to the area of $\frac{M}{EI}$ diagram between those two points.

Where,

M = Bending Moment

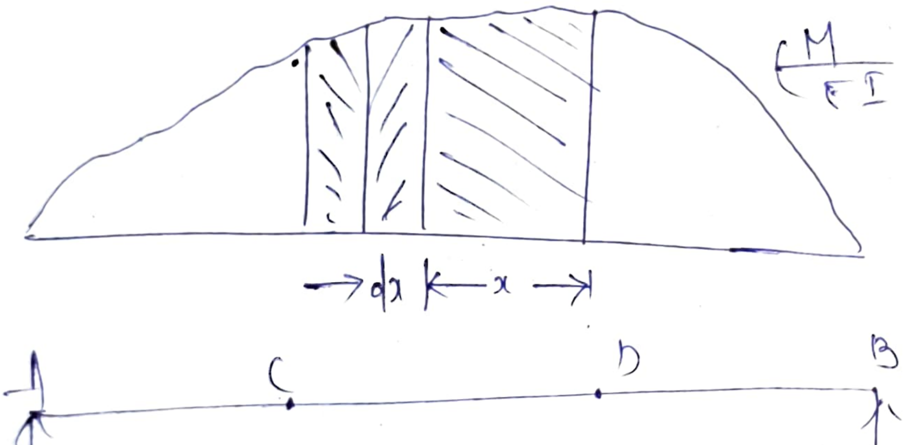
E = Young's Modulus

I = Moment of Inertia

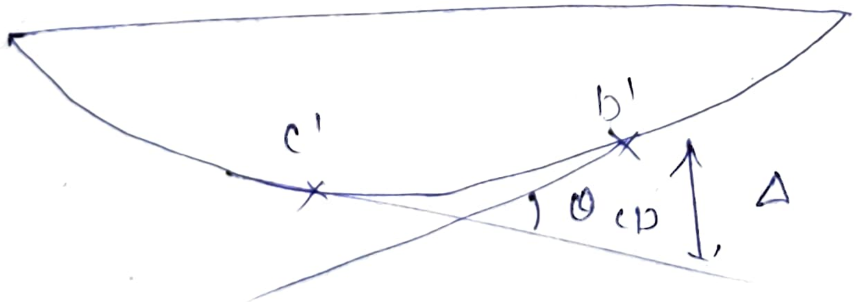
Consider the beam AB.
Let C & D be any two points of this beam.
The $\left(\frac{M}{EI}\right)$ diagram is also shown below figure.

According to this theorem, θ_{CD} which is the angle between the tangents at C and D is equal to the area of $\left(\frac{M}{EI}\right)$ diagram between C & D.

$$\boxed{\theta_{CD} = \int_C^D \left(\frac{M}{EI}\right) dx}$$



Beam with $\frac{M}{EI}$ diagram as load



Angle bend defl^n betn the tangents at C and D.

Thm. 2)

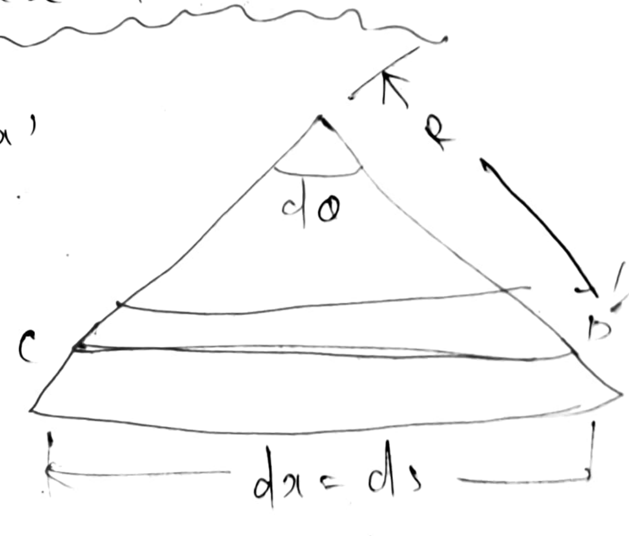
Deflection at a point in a beam in the direction ⊥r to its original straight line position measured from the tangent to the elastic curve at another point is given by the moment of $\left(\frac{M}{EI}\right)$ diagram about the pt. where the defln is required.

Δ, the vertical (⊥r to the horizontal position of AB) deflection at pt. D' from the tangent to the elastic curve at C' is given by the moment of $\left(\frac{M}{EI}\right)$ diagram between C and D about the point D.

$$\boxed{\Delta = \int_c^x \left(\frac{M}{EI}\right) x \cdot dx}$$

Derivation of Moment Area Theorems:—

the elemental length 'dx' to an enlarged scale.



(Deformed shape of elemental length)

Let,

R = radius of curvature

Flexure formula,

$$\boxed{\frac{M}{I} = \frac{E}{R}}$$

$Rd\theta = ds = dx$ = axial deformations are considered negligible.

$$R = \frac{dx}{d\theta}$$

Substituting the value of R in flexure eq^n we get

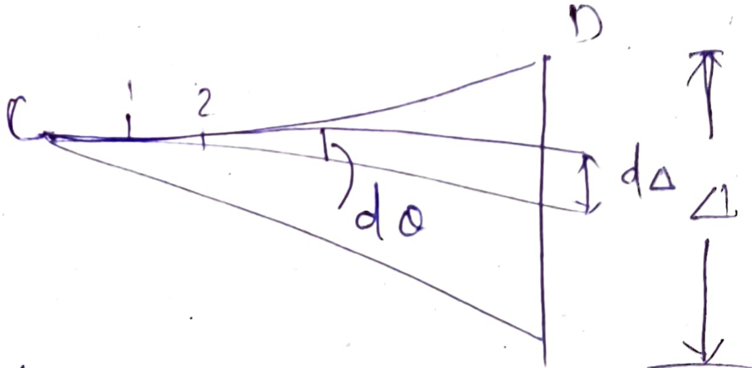
$$\frac{M}{I} = \frac{E}{(dx/d\theta)}$$

$$\Rightarrow d\theta = \left(\frac{M}{EI}\right) dx$$

$$\Rightarrow \theta_{CD} = \int_C^D d\theta = \int_C^D \left(\frac{M}{EI}\right) dx \quad \text{(proved)}$$

Thm II

Now consider portion CD is blown upto an enlarged scale.



(Enlarged view of portion CD)

Let the change of slope in elemental length dx be '$d\theta$'. Distance of elemental length from D is x.

Deflection, $d\Delta = x\,d\theta$.

$$\Rightarrow d\Delta = x\left(\frac{M}{EI}\right) dx$$

$$\Rightarrow \Delta = \int_C^D \left(\frac{M}{EI}\right) x\, dx \quad \text{(proved)}$$

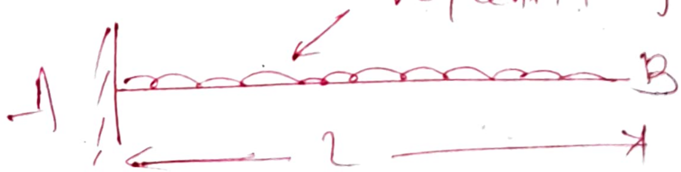
Sign convention in the Moment Area Method Applied to beams:—

1. Sagging moment area is positive, which means that, the tangent at D makes an anticlockwise angle with tangent at C.
2. The moment of +ve moment gives raise to positive deflection, which implies that the deflected position of a point (D) is above the tangent drawn at the other point C:

Ex:- 2.1

Determine the rotation and deflection at the free end of the cantilever beam subjected to uniformly distributed load over an entire span.

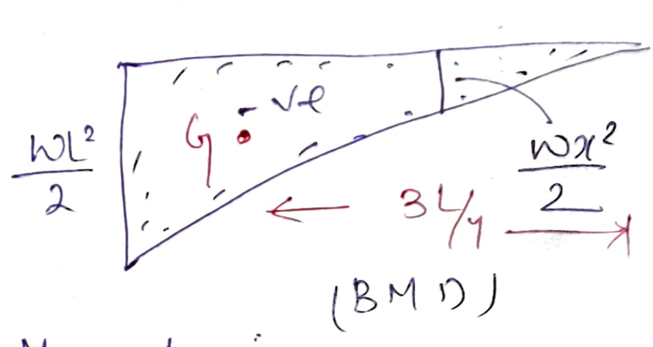
w/unit length



Soln:—

Rotation = θ = ?

Deflection = Δ = ?



$\theta_{BA} = \theta_B - \cancel{\theta_A} \to 0$

$= \theta_B = \int_0^L \left(\frac{M}{EI}\right) dx$

$\Rightarrow \theta_B = \int_0^L \frac{(-wx^2/2)}{EI} dx$

$$\Rightarrow \theta_B = \frac{-w}{2EI}\int_0^L x^2\,dx$$

$$= \frac{-w}{2EI}\left[\frac{x^3}{3}\right]_0^L$$

$$= \frac{-w}{2EI}\times\frac{1}{3}(L^3-0)$$

$$= \frac{-wL^3}{6EI}$$

$$\Rightarrow \theta_B = \frac{wL^3}{6EI}$$ (clockwise with tangent at A

Δ_B = defln of B w.r.t tangent at A

= vertical defln, since, tangent at A is horizontal

$$= \int_0^L \left(\frac{M}{EI}\right) x\,dx$$

$$= \int_0^L \frac{(-wx^2/2)}{EI}\,x\,dx$$

$$= \frac{-w}{2EI}\int_0^L x^3\,dx$$

$$= \frac{-w}{2EI}\left[\frac{x^4}{4}\right]_0^L$$

$$= \frac{-w}{2EI}\times\frac{1}{4}(L^4-0)$$

$$\Rightarrow \Delta_B = \frac{-wL^4}{8EI}$$

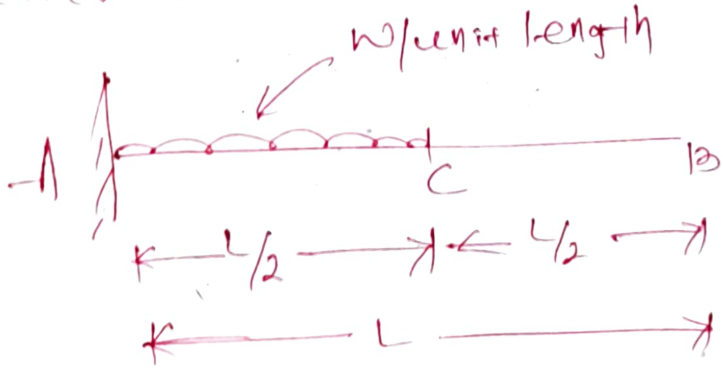
$$= \frac{wL^4}{8EI} \text{ (downward)}$$

* Area of such a parabolic curve $= \frac{1}{3} \times L \times$ ordinate at the end and its centre of gravity is at a distance $\frac{3L}{4}$ from the end where the value is zero.

Ex!- 2.2

Find the rotation and deflection at the free end in the cantilever beam.

Soln! →

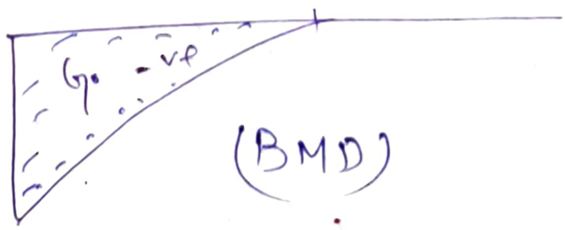


Maxm ordinate =

$$\frac{wL}{4} \times \frac{L}{4} = \frac{wL^2}{8}$$

C.G from the pt. C is at a distance =

$$\frac{3}{4} \times \frac{L}{2} = \frac{3}{8}L$$



(BMD)

$$\text{Area of B.M.D} = \frac{1}{3} \times \frac{L}{2} \times \frac{wL^2}{8} = \frac{wL^3}{48}$$

$$\theta_{BA} = \theta_B - \theta_A$$

$$= \theta_B \quad (\because \theta_A = 0)$$

θ_B = Area of $\left(\frac{M}{EI}\right)$ diagram betn A and B

$$= \left[-\frac{WL^3}{48}\right] \times \frac{1}{EI}$$

$$= \frac{-WL^3}{48EI}$$

$$= \frac{WL^3}{48EI}, \text{ (clockwise)}$$

Tangent at A is horizontal.

Vertical defln at B = Moment of $\left(\frac{M}{EI}\right)$ diagram about B

$$= \left(\frac{-WL^3}{48EI}\right) \times \left(\frac{3}{8}L + \frac{L}{2}\right)$$

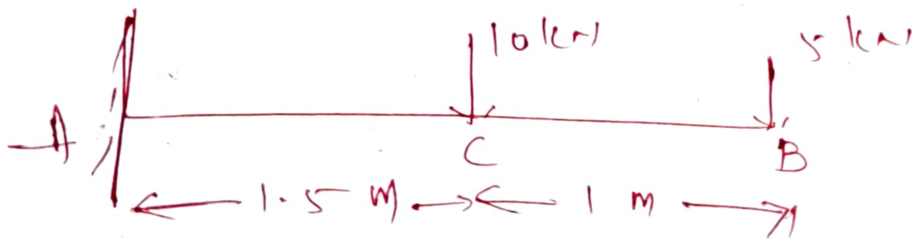
$$= \frac{-7WL^4}{384EI}$$

$$= \frac{7WL^4}{384EI} \text{ (downward)}$$

Ex:- 2.3

Determine the slope and defln at the free end of a cantilever beam by moment area method. (take $EI = 4000\ kN\,m^2$)

Ans:-

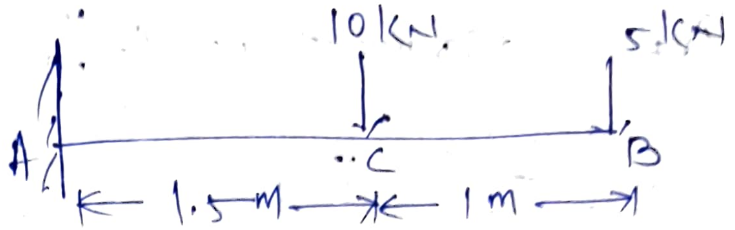


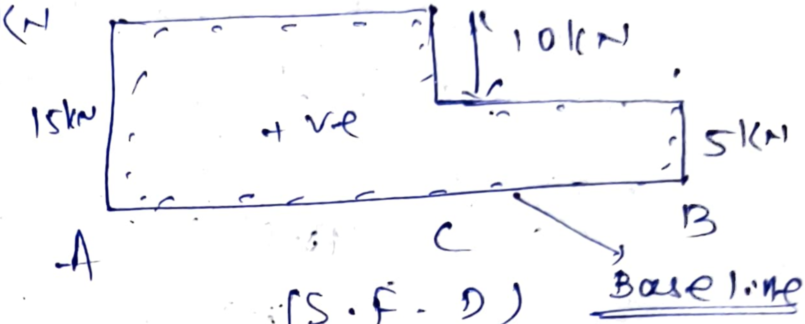
Sol^n:-

S.F at B, $S.F_B = 5\,kN$

S.F at C, $S.F_C = 5+10$
$= 15\,kN$

S.F at A, $S.F_A = 5+10 = 15\,kN$



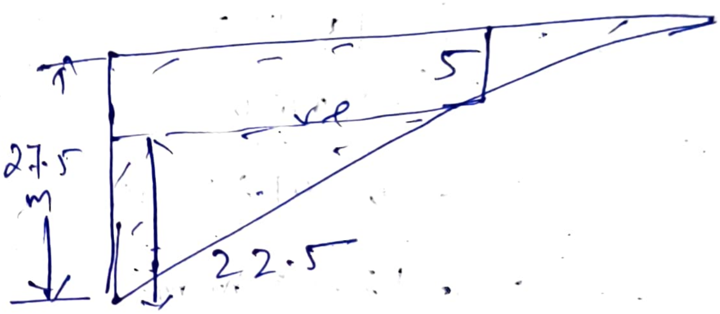


B.M at B, $BM_B = 0$

B.M at C, $M_C =$

$-5\,kN \times 1\,m$

$= -5\,kN\text{-}m$



B.M at A, $M_A =$

$-5 \times (1+1.5)$

-10×1.5

$= -\cancel{5} \times \frac{2.5}{\cancel{10}_2} - 15$

$= -12.5 - 15 = 27.5$
kN-m

$\theta_B = \frac{\text{Area of BMD}}{EI}$

$= \frac{1}{EI}\left(\frac{1}{2}\times 1 \times 5 + 5 \times 1.5 + \frac{1}{2} \times 22.5 \times 1.5\right)$

$= \frac{1}{EI}(26.875) = \frac{-26.875}{4000}$

$= -6.71875 \times 10^{-3}$ radians

$= 6.71875 \times 10^{-3}$ radians (clockwise)

Δ_B = Moment of $\left(\frac{M}{EI}\right)$ diagram about B

$$= \frac{1}{EI}\left[\frac{1}{2}\times1\times5\times\frac{2}{3} + 5\times1.5\times(1+0.75) + \frac{1}{2}\times22.5\times1.5\times(1+1)\right]$$

$$= -\frac{48.542}{EI}$$

$$= \frac{-48.542}{4000}$$

$= -0.01213$ m

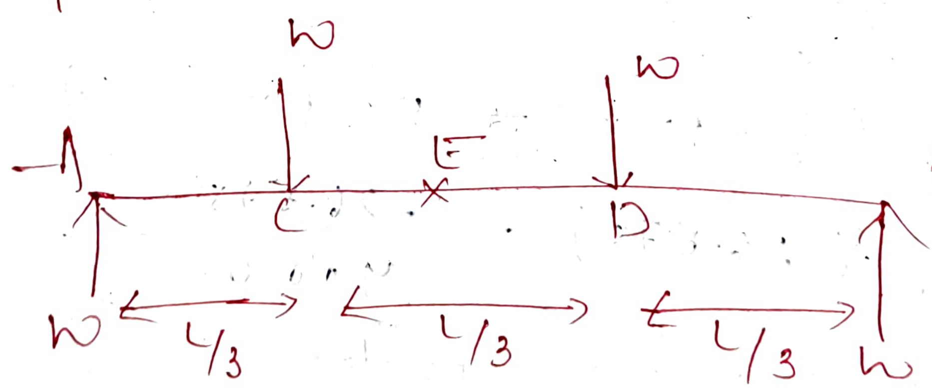
$= -12.13$ mm

$= 12.13$ mm (downward)

Ex-2.4

Determine the rotation at supports and deflection at mid-span under the loads in the simply supported beam.

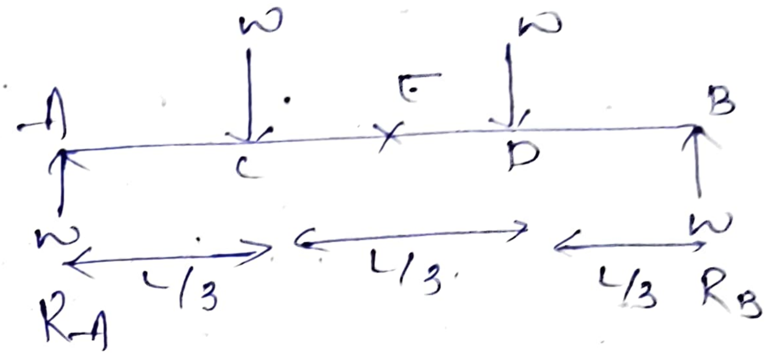
Ans:→



Taking moments of the force about A,

$R_B \times \left(\frac{L}{3}+\frac{L}{3}+\frac{L}{3}\right)$

$= W\left(\frac{L}{3}+\frac{L}{3}\right) + W\left(\frac{L}{3}\right)$

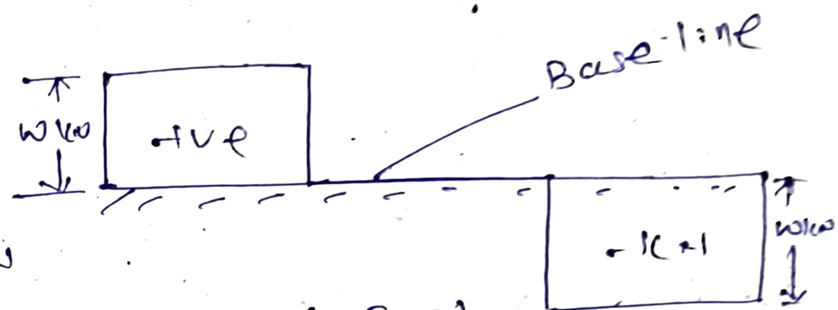


$\Rightarrow R_B \times L = \frac{2WL}{3} + \frac{WL}{3}$

$\Rightarrow R_B \times L = \frac{3WL}{3}$

$\Rightarrow R_B = W$ kN

$R_A = (W+W) - W = W$ kN



(S.F.D)

S.F.D:—

S.F at A, $F_A = +R_A = W$ kN

S.F betn A & C, is const and equal to W kN.

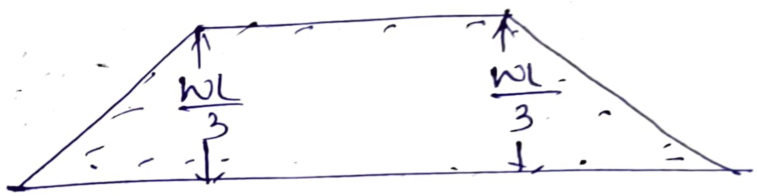
S.F at C, $F_C = W - W = 0$

S.F betn C & D is const and equal to 0 kN.

S.F at D, $F_D = 0 - W = -W$ kN

S.F betn D & B is const. and equal to $-W$ kN.

S.F at B, $F_B = -W$ kN.



(B.M.D)

B.M.D:—

$M_A = 0$

$M_C = R_A \times \frac{L}{3} = \frac{WL}{3}$

$M_D = R_A \times \frac{2L}{3} - W \times \frac{L}{3}$

$= \frac{2WL}{3} - \frac{WL}{3} = \frac{WL}{3}$

$M_B = 0$

* In this case, the BMD is symmetry, so the slope at E is zero.
- The tangent at E is horizontal.
- It is convenient to work out rotations and defln w.r.t the tangent at E.

θ_A = Area of $\left(\frac{M}{EI}\right)$ diagram betn A and E

$$= \frac{1}{EI}\left(\frac{1}{2} \times \frac{WL}{3} \times \frac{L}{3} + \frac{1}{2} \times \frac{WL}{3} \times \frac{L}{3}\right)$$

$$= \frac{1}{EI}\left(\frac{WL^2}{18} + \frac{WL^2}{18}\right)$$

$$= \frac{1}{EI} \times \frac{2WL^2}{18}$$

$$= \frac{WL^2}{9EI}$$

Defln of E, w.r.t A = Defln of A. w.r.t. E

= Moment of $\left(\frac{M}{EI}\right)$ diagram betn AE about A

$$= \frac{1}{EI}\left[\frac{1}{2} \times \frac{L}{3} \times \frac{WL}{3} \times \frac{2}{3} \times \frac{L}{3} + \frac{1}{2} \times \frac{L}{3} \times \frac{WL}{3}\left(\frac{L}{3} + \frac{L}{12}\right)\right]$$

$$= \frac{WL^3}{EI} \times \left(\frac{1}{81} + \frac{1}{18} \times \frac{5}{12}\right)$$

$$= \frac{23\,WL^3}{648\,EI}$$

Defln of C above E = Moment of $\frac{M}{EI}$ dig. betn CE about C

$$= \frac{1}{EI} \times \frac{L}{6} \times \frac{WL}{3} \times \frac{L}{12}$$

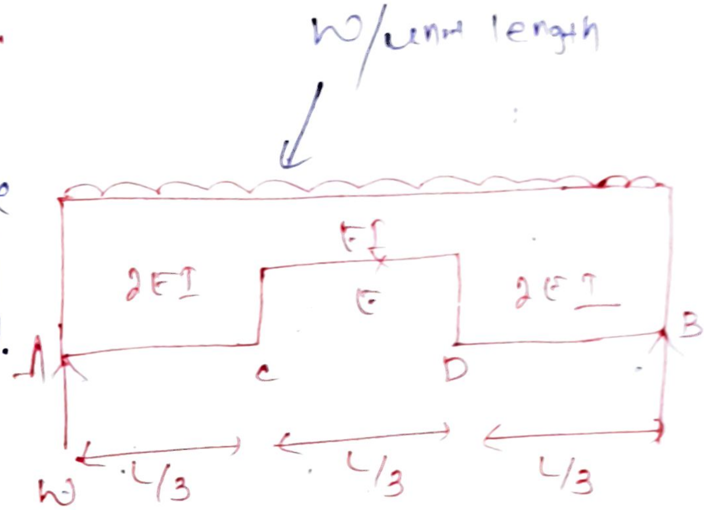
$$= \frac{WL^3}{216\,EI}$$

Defln of C below A $= \left(\frac{23}{648} - \frac{1}{216}\right)\frac{WL^3}{EI} = \frac{5}{162}\frac{WL^3}{EI}$

Due to symmetry, defln of D below B $= \frac{5}{162}\frac{WL^3}{EI}$

Ex: 2.5

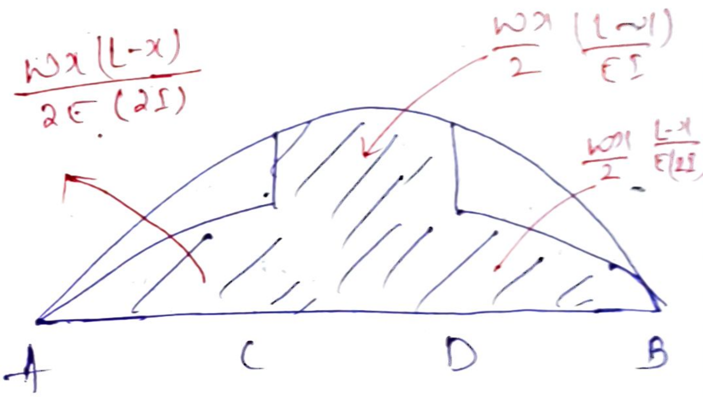
Determine the slope at A, defl^n at C and mid-span E in the beam.



Sol^n:-

Due to symmetry, the slope at mid-span E is zero and tangent at this pt. is horizontal.

- slope at A w.r.t to the horizontal axis is the area of $\left(\frac{M}{EI}\right)$ diagram betn A and E and vertical defl^n of E is the moment of $\left(\frac{M}{EI}\right)$ dig. betn A and E. about E (i.e., we calculate upward defl^n of A from tangent at E.)



$\left(\frac{M}{EI}\text{ diagram}\right)$

- the B.M at any pt. at a distance x from end A is given by:- $M_x = \frac{w}{2}(Lx - x^2)$

- Elemental length dx is first considered for finding the area of moment dig. at the required pt. and then it is integrated with in the appropriate limits.

- the $\left(\frac{M}{EI}\right)$ dig. for this,

θ_A = Area of $\left(\frac{M}{EI}\right)$ dig. betn A and E

= Area of $\left(\frac{M}{EI}\right)$ dig. betn A and C + Area of $\left(\frac{M}{EI}\right)$ dig. betn C and E.

$$= \int_0^{L/3} \frac{M_x}{2EI} + \int_{L/3}^{L/2} \frac{M_x}{EI}\, dx$$

$$= \int_0^{L/3} \frac{w}{4EI}(Lx - x^2)\, dx + \int_{L/3}^{L/2} \frac{w}{2EI}(Lx - x^2)\, dx$$

$$= \frac{w}{4EI}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^{L/3} + \frac{w}{2EI}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_{L/3}^{L/2}$$

$$= \frac{wL^3}{4EI}\left(\frac{1}{18} - \frac{1}{81}\right) + \frac{wL^3}{2EI}\left(\frac{1}{8} - \frac{1}{24} - \frac{1}{18} + \frac{1}{81}\right)$$

$$= \frac{5wL^3}{162EI}$$ slope at C side. this is the area of $\left(\frac{M}{EI}\right)$ dig. betn C and E.

$\Delta_E = \Delta_A$ w.r.t horizontal axis at E

= moment of $\left(\frac{M}{EI}\right)$ dig. betn A and E about A.

$$= \int_0^{L/3} \left(\frac{1}{2EI}\right)\left(\frac{w}{2}\right)(Lx - x^2)\, x\, dx$$

$$+ \int_{L/3}^{L/2} \left(\frac{1}{EI}\right)\left(\frac{w}{2}\right)(Lx - x^2)\, x\, dx$$

$$= \frac{w}{4EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_0^{L/3} + \frac{w}{2EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_{L/3}^{L/2}$$

$$= \frac{wL^4}{4EI}\left(\frac{1}{81} - \frac{1}{81\times 4}\right) + \frac{wL^4}{2EI}\left(\frac{1}{24} - \frac{1}{64} - \frac{1}{81} + \frac{1}{81\times 4}\right)$$

$$= 0.010706 \times \frac{wL^4}{EI}$$

$$= \int_0^{L/3} \frac{M_x}{2EI} + \int_{L/3}^{L/2} \frac{M_x}{EI}\, dx$$

$$= \int_0^{L/3} \frac{w}{4EI}(Lx - x^2)\, dx + \int_{L/3}^{L/2} \frac{w}{2EI}(Lx - x^2)\, dx$$

$$= \frac{w}{4EI}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^{L/3} + \frac{w}{2EI}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_{L/3}^{L/2}$$

$$= \frac{wL^3}{4EI}\left(\frac{1}{18} - \frac{1}{81}\right) + \frac{wL^3}{2EI}\left(\frac{1}{8} - \frac{1}{24} - \frac{1}{18} + \frac{1}{81}\right)$$

$$= \frac{5wL^3}{162EI}$$

slope at C site. this is the area of $\left(\frac{M}{EI}\right)$ dig. betn C and E.

$\Delta_E = \Delta_A$ w.r.t horizontal axis at E

= moment of $\left(\frac{M}{EI}\right)$ dig. betn A and E about A.

$$= \int_0^{L/3} \left(\frac{1}{2EI}\right)\left(\frac{w}{2}\right)(Lx - x^2)\, x \cdot dx$$

$$+ \int_{L/3}^{L/2} \left(\frac{1}{EI}\right)\left(\frac{w}{2}\right)(Lx - x^2)\, x\, dx$$

$$= \frac{w}{4EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_0^{L/3} + \frac{w}{2EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_{L/3}^{L/2}$$

$$= \frac{wL^4}{4EI}\left(\frac{1}{81} - \frac{1}{81 \times 4}\right) + \frac{wL^4}{2EI}\left(\frac{1}{24} - \frac{1}{64} - \frac{1}{81} + \frac{1}{81 \times 4}\right)$$

$$= 0.010706 \times \frac{wL^4}{EI}$$

Defl^n at C :-

At C, 1st defl^n above the tangent at E will be found by moment area method and then, it is substracted from the defl^n of E to get the defl^n of C below the original axis.

Δ_C = moment of $\left(\frac{M}{EI}\right)$ dig. bet^n C and E about C.

$$= \int_{L/3}^{L/2} \left(\frac{1}{EI}\right)\left(\frac{w}{2}\right)(Lx - x^2)\left(x - \frac{L}{3}\right)dx$$

$$= \frac{w}{2EI}\int_{L/3}^{L/2}\left(Lx^2 - x^3 - \frac{L^2}{3}x + \frac{L}{3}x^2\right)dx$$

$$= \frac{w}{2EI}\int_{L/3}^{L/2}\left(\frac{4L}{3}x^2 - x^3 - \frac{L^2}{3}x\right)dx$$

$$= \frac{w}{2EI}\left[\frac{4L}{3}\times\frac{x^3}{3} - \frac{x^4}{4} - \frac{L^2}{6}\times x^2\right]_{L/3}^{L/2}$$

$$= \frac{wL^4}{2EI}\left(\frac{4}{9\times 8} - \frac{1}{64} - \frac{1}{24} - \frac{4}{243} + \frac{1}{81\times 4} + \frac{1}{6\times 9}\right)$$

$$= 1.703961\times 10^{-3}\left(\frac{wL^4}{EI}\right)$$

$$\Delta_C = \Delta_E - \Delta_C$$

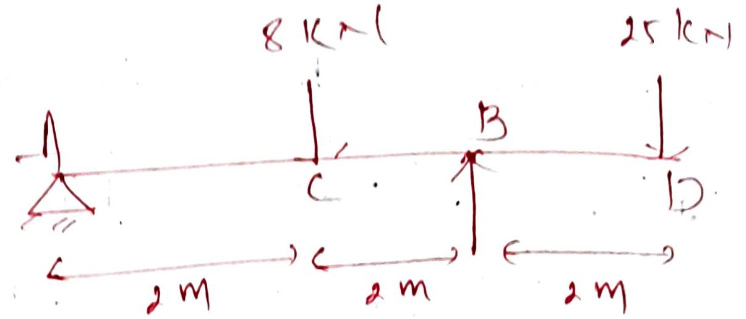
$$= (0.0107 06 - 1.703961\times 10^{-3})\frac{wL^4}{EI}$$

$$= 0.00900205\left(\frac{wL^4}{EI}\right)$$

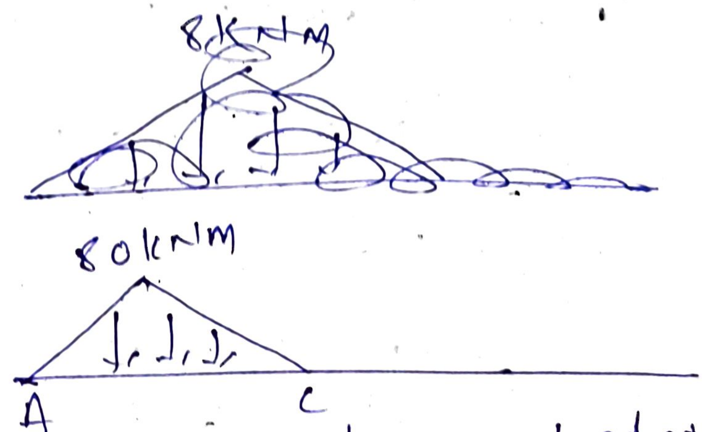
Ex:- 2.6

Determine the slope and defl^n at the end of the beam. EI is const. throughout.

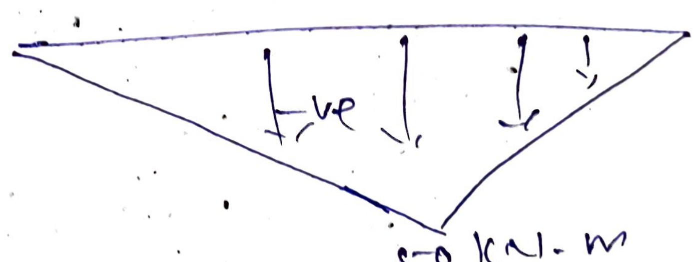
Soln:-



B.M.D due to load at C and that due to the load at D.



BMD due to load at C



BMD due to load at D

slope at B = $\frac{AA_1}{AB}$

where,

AA_1 = Defl^n at A w.r.t the tangent at B

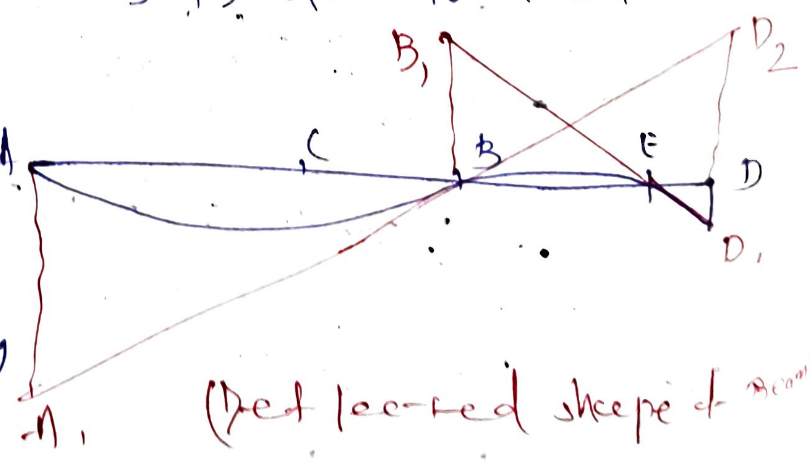
= Moment ($\frac{M}{EI}$) dig. betn A and B about A

$= \frac{1}{EI}\left(\frac{1}{2}\times 80\times 4\times 2 - \frac{1}{2}\times 50\times 4\times \frac{8}{3}\right)$

$= \frac{53.333}{EI}$

$\theta_B = \frac{53.333}{EI}\times\frac{1}{4} = \frac{13.333}{EI}$

By similar Δ principle,



(Deflected shape of Beam)

$DD_2 = \frac{AA_1}{2}$

$= \frac{26.667}{EI}$ (anticlockwise)

D_1D_2 = Defl^n of pt. D w.r.t the tangent at B

= Moment ($\frac{M}{EI}$) dig. betn B and D about D

$= \frac{1}{EI}\times\frac{1}{2}\times 50\times 2\times\frac{4}{3}$

$\Rightarrow D_1 D_2 = \dfrac{-66.667}{EI}$

$= \dfrac{66.667}{EI}$ (downward)

vertical defln at D,

$DD_1 = \dfrac{66.667}{EI} - \dfrac{26.667}{EI} = \dfrac{40}{EI}$

to find slope at D,

BB_1 = Defln of B. w.r.t the tangent at D

= Moment ($\frac{M}{EI}$) dig. betn B and D about B

$= \dfrac{1}{EI}\left(-\dfrac{1}{2} \times 50 \times 2 \times \dfrac{2}{3}\right)$

$= \dfrac{-33.333}{EI}$

let E be the intersection of B_1D_1 and BD,

let $DE = x$

$\dfrac{DE}{BE} = \dfrac{x}{2-x}$

$\dfrac{DE}{BE} = \dfrac{DD_1}{BB_1}$

$\therefore \dfrac{x}{2-x} = \dfrac{40}{33.33} = 1.2$

$\Rightarrow x = (2-x) \times 1.2$

$\Rightarrow x = 1.091$

slope at D = $\frac{DD_1}{x}$

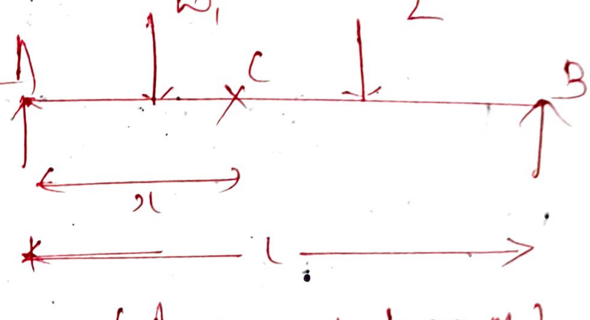
$\Rightarrow \theta_D = \frac{40}{EI} \times \frac{1}{1.091}$

$\Rightarrow \theta_D = \frac{36.667}{EI}$

* Moment of area method can be applied to any type of determinate beam, but it can be used conveniently only if the tangent at one of the points in the beam is horizontal, i.e, if there is a fixed end or a point of symmetry.

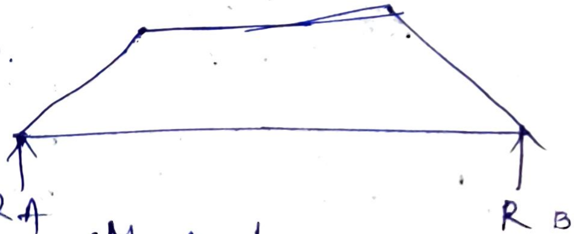
Conjugate Beam theorems :-

* these theorems can be derived from moment area theorems and are very useful in finding the deflⁿ even if there is no point in the beam where the slope is zero.



(A typical beam)

- consider the simply supported beam.

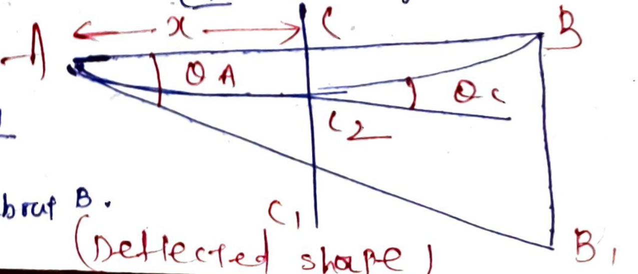


$\left(\frac{M}{EI}\right)$ diagram

$\theta_C = \theta_A$ - Area of $\left(\frac{M}{EI}\right)$ dig. betⁿ A and C.

$\theta_A = \frac{BB_1}{AB}$

$= \left(\frac{1}{L}\right)$ moment of area of $\left(\frac{M}{EI}\right)$ dig. betⁿ A and B about B.



(Deflected shape)

$$\theta_C = \frac{\text{Moment of } \left(\frac{M}{EI}\right) \text{ dig. about } B}{L}$$

$- \text{Area of } \left(\frac{M}{EI}\right) \text{ dig. betn } A \text{ and } C.$

Defln at C $= CC_2 = CC_1 - C_2C_1$

$= x_C \theta_A$ - Defln of C w.r.t. Tangent at A

$$= \frac{x_C \times \text{Moment of } \left(\frac{M}{EI}\right) \text{ dig. betn } A \text{ \& } B \text{ about } B}{L}$$

$- \text{Area of } \left(\frac{M}{EI}\right) \text{ dig. betn } A \text{ and } C \text{ about } C.$

Consider an imaginary beam of same span, loaded with $\left(\frac{M}{EI}\right)$ diagram.

Reaction at A, $R_A' = \dfrac{\text{Moment of the load about } B}{L}$

$$= \frac{\text{Moment of } \left(\frac{M}{EI}\right) \text{ dig. betn } A \text{ \& } B \text{ about } B}{L}$$

S.F at C $= R_A'$ - load betn A & C

$$= \frac{\text{Moment of } \left(\frac{M}{EI}\right) \text{ dig. betn } A \text{ \& } B \text{ about } B}{L}$$

$- \text{Area of } \left(\frac{M}{EI}\right) \text{ dig. betn } A \text{ \& } C.$

* θ_C is the given beam is equal to the S.F in the beam loaded with $\left(\frac{M}{EI}\right)$ dig. Similarly, it can be observed that the defln of C, given by CC_2, is equal to the B.M in the imaginary beam loaded with $\left(\frac{M}{EI}\right)$ diagram.

Thm 1

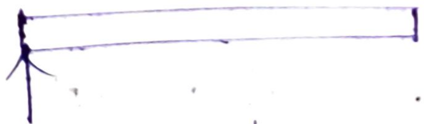
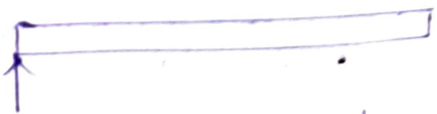
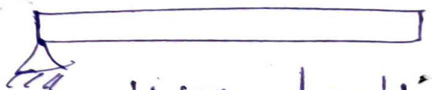
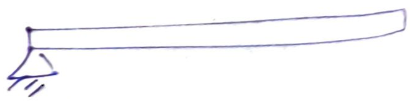
The rotation at a point in a beam is equal to the shear force in the conjugate beam.

Thm 2

The defl^n in a beam is equal to the B.M in the conjugate beam.

Conjugate beam

It is an imaginary beam of same span as the original beam loaded with $\left(\frac{M}{EI}\right)$ dig. of the original beam, such that the S.F and B.M at a sect^n will represent the rotation and defl^n at that section in the original beam.

Original Beam	Conjugate Beam
	
(Simply supported or Roller end)	(Simply supported or Roller end)
$\theta \neq 0$	$F \neq 0$
$\Delta = 0$	$M = 0$
	
(Hinged end)	(Hinged end)
$\theta \neq 0$	$F \neq 0$
$\Delta = 0$	$M = 0$

fixed end

$\theta = 0$

$\Delta = 0$

free end

$F = 0$

$M = 0$

free end

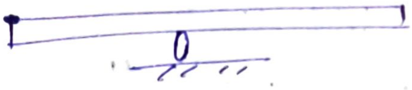
$\theta \neq 0$

$\Delta \neq 0$

fixed end

$F \neq 0$

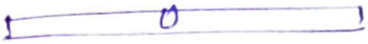
$M \neq 0$



Internal support

$\theta \neq 0$ and continuous

$\Delta = 0$



Internal hinge

$F \neq 0$ and continuous

$M = 0$

Internal Hinge

$\theta \neq 0$ and discontinuous

$\Delta \neq 0$

Internal support

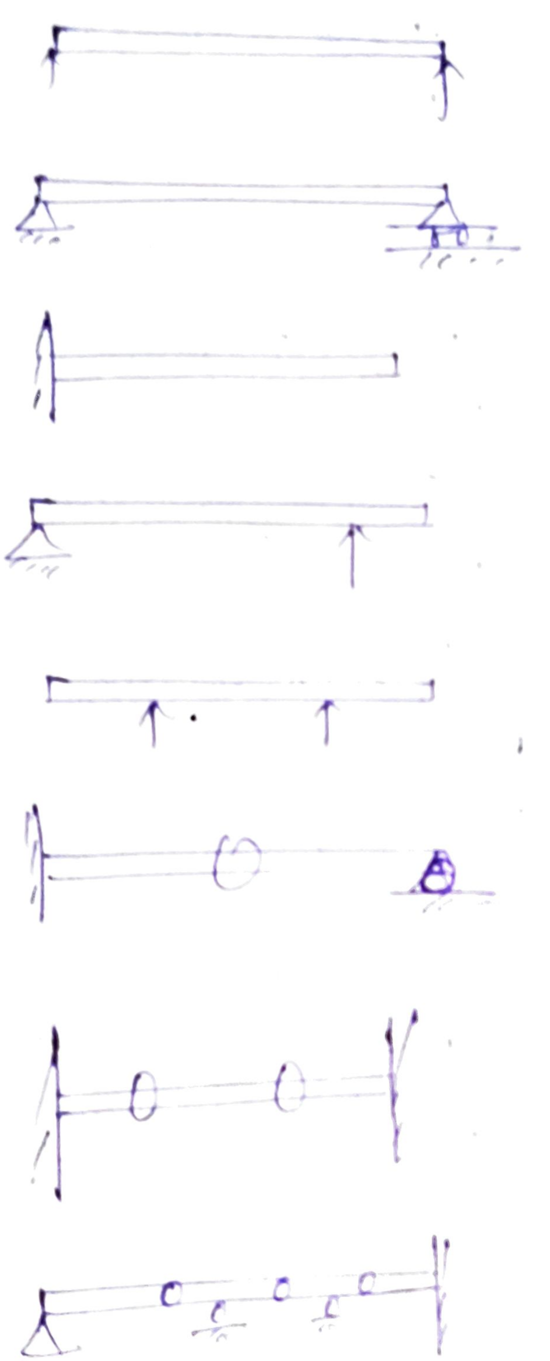
$F \neq 0$ and discontinuous

$M \neq 0$

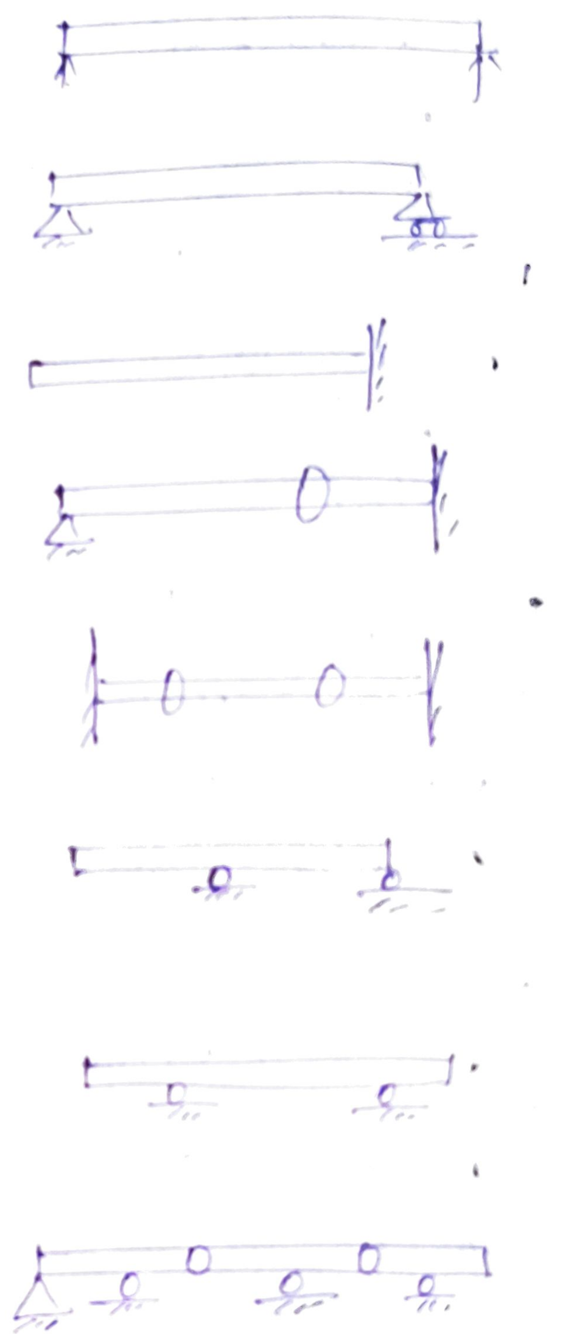
Sign convention!

- Sagging moment is positive moment.
- Left-side upward force or right-side downward force ($\uparrow \downarrow$) gives +ve shear.
- +ve shear gives clockwise rotation and +ve moment gives downward defl^n.

Original beams



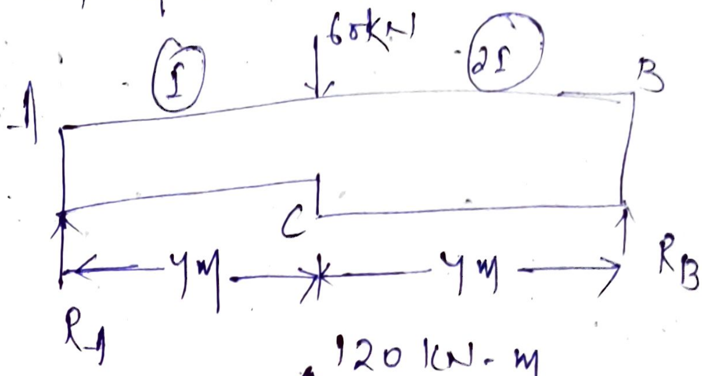
Conjugate beams



Q.2.7

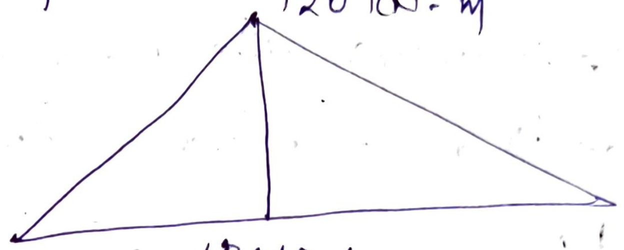
Determine θ_A, θ_B, θ_C and defl^n Δ.

Ans:—



Load bet^n A and C:→

$$\frac{1}{2} \times \frac{120}{EI} \times 4 = \frac{240}{EI}$$



(BMD)

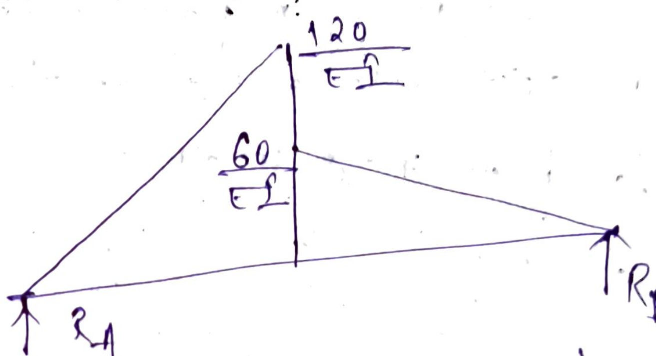
$$C.G = 4 + \frac{4}{3} = 5.333\ m$$
from B.

Load bet^n C and B:→

$$\frac{1}{2} \times \frac{60}{EI} \times 4 = \frac{120}{EI}$$

C.G at a distance :—

$$\left(\frac{2}{3} \times 4\right) = \frac{8}{3}\ m \text{ from B.}$$



(Conjugate Beam)

$$\Sigma M_B = 0$$

$$\Rightarrow R_A' \times 8 = \frac{240}{EI} \times 5.333 + \frac{120}{EI} \times \frac{8}{3}$$

$$\Rightarrow R_A' = \frac{200}{EI} \text{ units.}$$

$$R_B' = \frac{240}{EI} + \frac{120}{EI} - \frac{200}{EI} = \frac{160}{EI} \text{ units.}$$

$$\theta_A = \text{S.F at A} = R_B' = \frac{200}{EI} \text{ (clockwise)}$$

$$\theta_B = \text{S.F at B} = -R_B' = \frac{-160}{EI} = \frac{160}{EI} \text{ radians}$$
(anticlockwise)

$$\theta_C = \text{S.F at C}$$
$$= \frac{200}{EI} - \frac{240}{EI} = \frac{-40}{EI} \text{ units}$$
$$= \frac{40}{EI} \text{ radians. (anticlockwise)}$$

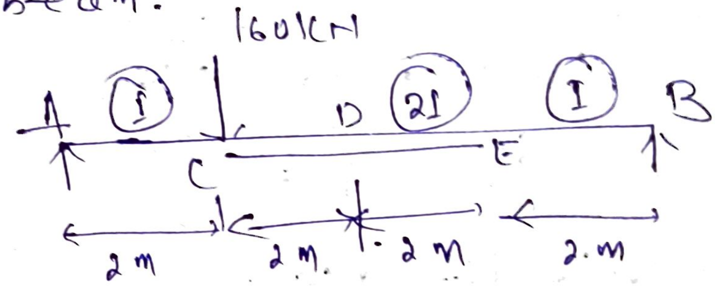
Δ_C = Moment at C

$$= \left(\frac{200}{EI} \times 4\right) - \left(\frac{240}{EI} \times \frac{4}{3}\right)$$

$$= \frac{480}{EI} \quad \text{(downward)}$$

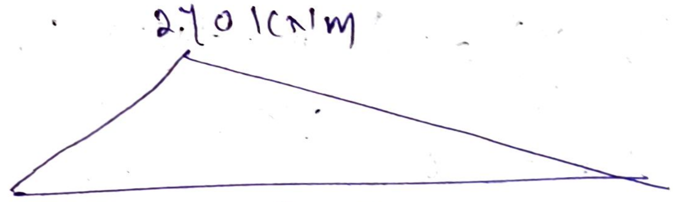
Ex. 2.8

Determine the rotations at A, B, C, D, E and deflection at C, D & E in the beam.



Ans:-

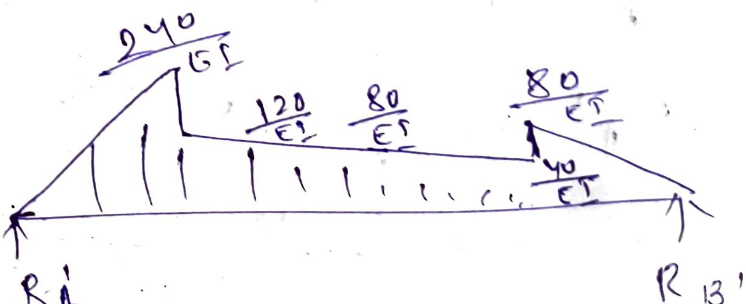
The load in portion CE is split into a triangular load of max intensity



(BMD)

$$= \frac{120}{EI} - \frac{40}{EI} = \frac{80}{EI}$$

udl of $\frac{40}{EI}$ per metre length.



(conjugate beam)

$$\Sigma M_B = 0$$

$$\Rightarrow R_A' \times 8 = \frac{1}{2} \times \frac{240}{EI} \times 2\left(6 + \frac{2}{3}\right)$$

$$+ \frac{1}{2} \times \frac{80}{EI} \times 4\left(2 + \frac{2}{3} \times 4\right)$$

$$+ \frac{40}{EI} \times 4 \times 4$$

$$+ \frac{1}{2} \times \frac{80}{EI} \times 2 \times \frac{4}{3}$$

$$\Rightarrow R_A' = \frac{386.667}{EI}$$

$R_B' = \text{Total load} - R_A'$

$= \left(\frac{1}{2} \times \frac{240}{EI} \times 2 + \frac{1}{2} \times \frac{80}{EI} \times 4 + \frac{40 \times 4}{EI} + \frac{1}{2} \times \frac{80}{EI} \times 2\right) - \frac{386.667}{EI}$

$= \frac{253.333}{EI}$

$\theta_A = F_A = R_A' = \frac{386.667}{EI}$ radians, anticlockwise

$\theta_B = F_B' = -R_B' = \frac{-253.333}{EI}$

$= \frac{253.333}{EI}$ radians, anticlockwise.

$\theta_C = F_C = \frac{386.667}{EI} - \frac{1}{2} \times \frac{240}{EI} \times 2 = \frac{146.667}{EI}$ radians (clockwise)

$\theta_{C'} = -R_B' + \frac{1}{2} \times \frac{80}{EI} \times 2$

$= \frac{-253.333}{EI} + \frac{80}{EI}$

$= \frac{-173.333}{EI}$

$= \frac{173.333}{EI}$ radian, clockwise.

Δ_C = Moment of 'C' in conjugate beam

$= \frac{386.667}{EI} \times 2 - \frac{1}{2} \times \frac{240}{EI} \times 2 \times \frac{2}{3}$

$= \frac{613.333}{EI}$, downward

Δ_E = Moment at E

$= \frac{253.333}{EI} \times 2 - \frac{1}{2} \times \frac{80}{EI} \times 2 \times \frac{2}{3}$

$= \frac{453.333}{EI}$, downward

Δ_D = Moment at D

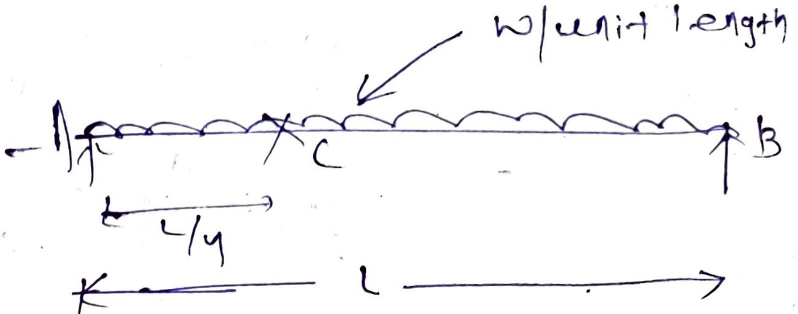
$= \frac{253.333}{EI} \times 4 - \frac{1}{2} \times \frac{80}{EI} \times 2 \times \left(2 + \frac{2}{3}\right)$

$- \frac{40}{EI} \times 2 \times 1 - \frac{1}{2} \times \left(\frac{80}{EI} - \frac{40}{EI}\right) \times 2 \times \frac{2}{3}$

$= \frac{693.333}{EI}$, (downward)

Ex:- 2.9

At any distance 'x' from support A,

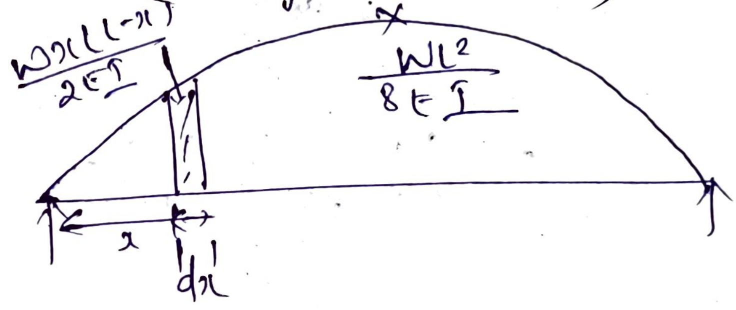


(Original Beam)

$$M = \frac{wx(L-x)}{2}$$

Load on conjugate beam

$$= \frac{1}{2EI} w(Lx - x^2)$$



Total load on conjugate beam = Area of $\left(\frac{M}{EI}\right)$ dia.

(conjugate beam)

$= \frac{2}{3} \times$ Area of rectangle enclosing the parabolic $\left(\frac{M}{EI}\right)$ dig.

$$= \frac{2}{3} \times \frac{WL^2}{8EI} \times L$$

$$= \frac{WL^3}{12EI}$$

$$R_A' = R_B' = \frac{1}{2} \times \text{Total load} = \frac{WL^3}{24EI}$$

θ_c = S.F at c in conjugate beam

$$= \frac{WL^3}{24EI} - \int_0^{L/4} \frac{w(Lx - x^2)}{2EI} dx$$

$$= \frac{WL^3}{24EI} - \frac{1}{2} \frac{w}{2EI} \left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^{L/4}$$

$$= \frac{WL^3}{EI} \left[\frac{1}{24} - \frac{1}{2}\left(\frac{1}{32} - \frac{1}{64 \times 3}\right)\right]$$

$$= \frac{WL^3}{EI} \left\{\frac{8 - \frac{1}{2}(6-1)}{192}\right\}$$

$$= \frac{5.5 WL^3}{192 EI}, \text{ clockwise}$$

Δ_C = Moment at 'c' in conjugate beam

$$= \frac{wL^3}{24EI} \times \frac{L}{4} - \int_0^{L/4} \frac{w(Lx - x^2)}{2EI} \left(\frac{L}{4} - x\right) dx$$

$$= \left(\frac{wL^3}{24EI} \times \frac{L}{4}\right) - \left(\frac{w}{8EI}\right) \int (L^2x - 5Lx^2 + 4x^3) dx$$

$$= \left(\frac{wL^3}{24EI} \times \frac{L}{4}\right) - \left(\frac{w}{8EI}\right) \left[\left(\frac{L^2x^2}{2}\right) - \left(\frac{5Lx^3}{3}\right) + x^4\right]_0^{L/4}$$

$$= \frac{wL^4}{EI}\left[\frac{1}{96} - \frac{1}{8}\left(\frac{1}{32} - \frac{5}{3 \times 64} + \frac{1}{256}\right)\right]$$

$$= 9.277 \times 10^{-3} \left(\frac{wL^4}{EI}\right)$$

Ex-2.10

Q_C = S.F in conjugate beam at c

$$= \left(\frac{15}{EI} + \frac{5}{EI}\right) \times \frac{1}{2} \times 2 + \frac{1}{3} \times \text{rectangle enclosing parabola}$$

$$= \frac{20}{EI} + \frac{1}{3} \times \frac{30}{EI} \times 2$$

$$= \frac{40}{EI}, \text{ clockwise}$$

Q_B = S.F at B in conjugate beam

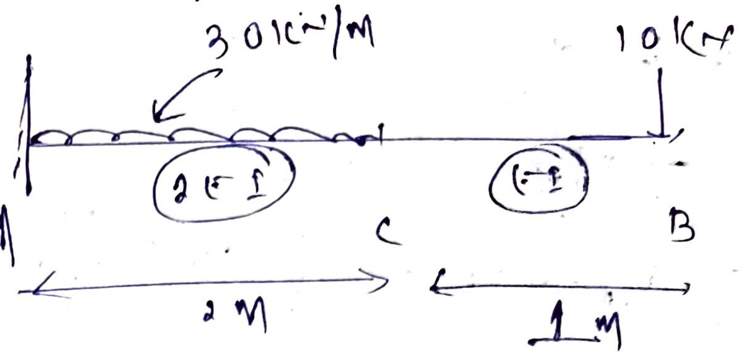
= S.F at c + load betn c and B

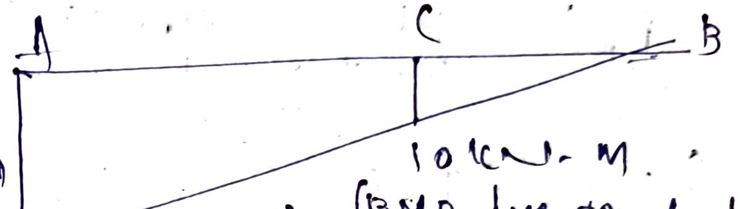
$$= \frac{40}{EI} + \frac{1}{2} \times \frac{10}{EI} \times 1$$

$$= \frac{45}{EI}, \text{ clockwise.}$$

Δ_C = B.M at C in conjugate beam

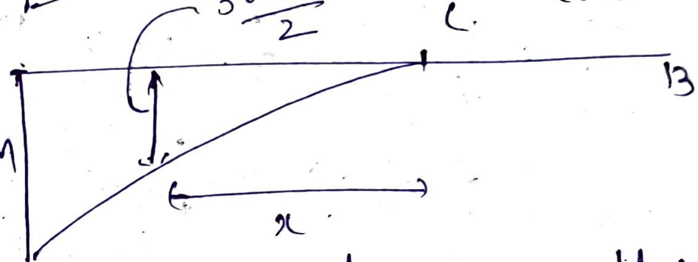
= B.M at C due to $\left(\frac{M}{EI}\right)$ d.g. and due to $\left(\frac{M}{EI}\right)$ d.g.





(BMD due to conc. load)

$$= \left[\frac{1}{2}\left(\frac{15}{EI}-\frac{5}{EI}\right)2\times\frac{4}{3}+\frac{5}{EI}\times 2\times 1\right]+\int_0^2 \frac{30x^2}{2EI}\,dx$$

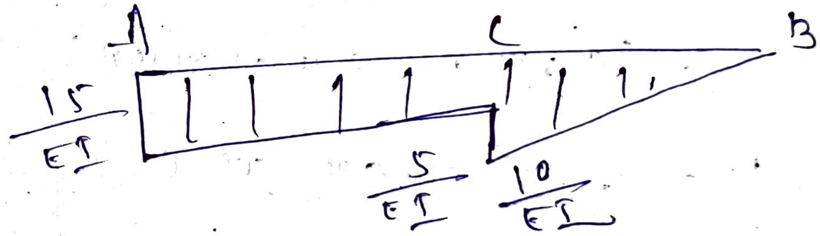


(BMD. due to udl)

$$= \frac{40}{3EI}+\frac{10}{EI}+\frac{15}{EI}\left[\frac{x^4}{4}\right]_0^2$$

$$= \frac{40}{3EI}+\frac{10}{EI}+\frac{60}{EI}$$

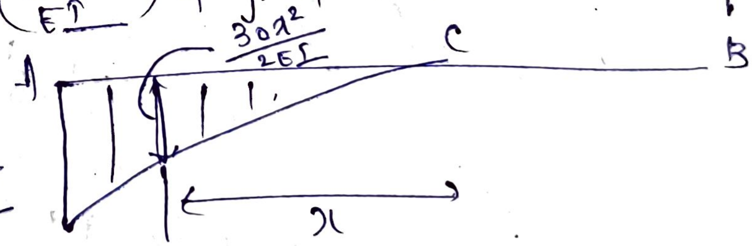
$$= \frac{83.33}{EI}, \text{ downward.}$$



Δ_B = B.M at B in the conjugate beam

$\left(\frac{M}{EI}\right)$ d.g. due to conc. load

$$= \left[\frac{1}{2}\left(\frac{15}{EI}-\frac{5}{EI}\right)\times 2\times\left(1+\frac{4}{3}\right)+\frac{5}{EI}\times 2\times 2+\frac{1}{2}\times\frac{10}{EI}\times 1\times\frac{2}{3}\right]$$



$\left(\frac{M}{EI}\right)$ d.g. due to udl.

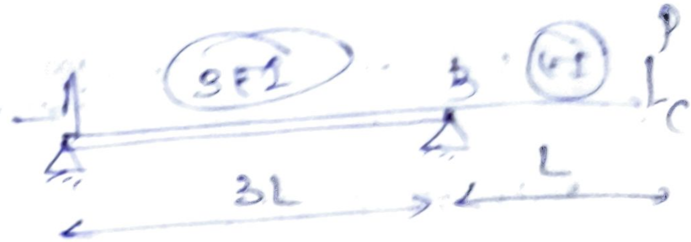
$$+\int_0^2 \frac{30x^2}{2EI}(x+1)\,dx$$

$$= \frac{46.667}{EI}+\frac{15}{EI}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^2$$

(conjugate beam)

$$= \frac{146.667}{EI}, \text{ (downward)}$$

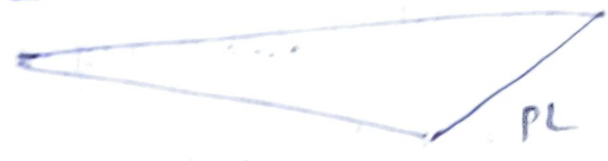
Ex 2.11



$\Sigma M_B = 0$

$\Rightarrow R_A' \times 3L = \frac{1}{2} \times 3L \times \frac{PL}{3EI} \times L$

$R_A' = \frac{PL^2}{6EI}$, downward

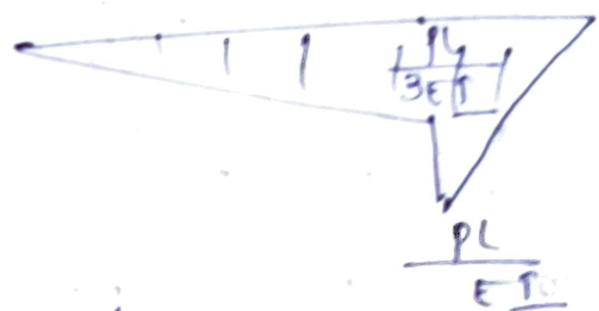


(BMD)

θ_C = S.F at C

$= \frac{-PL^2}{6EI} + \frac{1}{2} \times 3L \times \frac{PL}{3EI}$

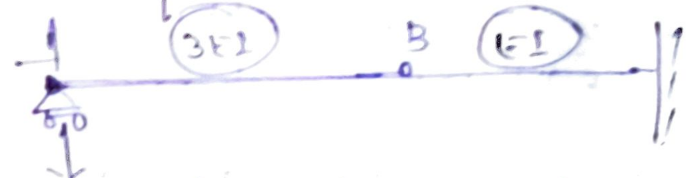
$+ \frac{1}{2} \times L \times \frac{PL}{EI}$



(load on conjugate beam)

$= 0.833 \left(\frac{PL^2}{EI}\right)$

(clockwise rotation)



R_A' (conjugate beam)

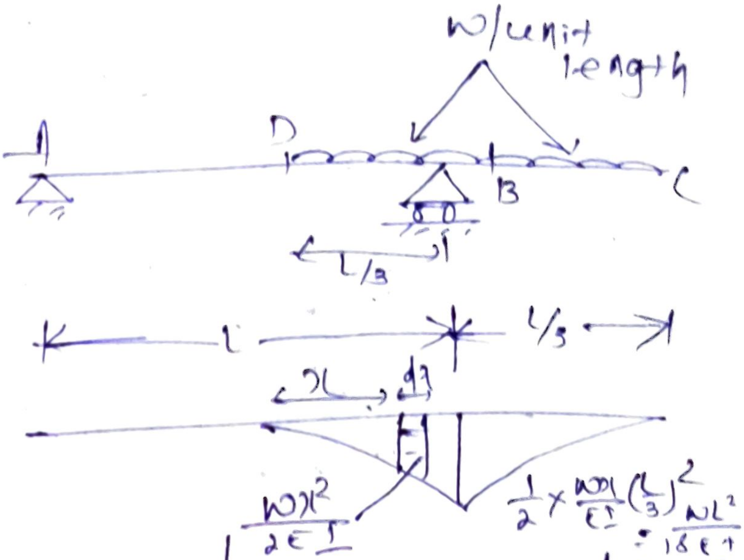
Δ_C = Moment at C

$= \frac{-PL^2}{6EI} \times 4L + \frac{1}{2} \times 3L \times \frac{PL}{3EI} \times 2L$

$+ \frac{1}{2} \times \frac{PL}{EI} \times L \times \frac{2L}{3}$

$= 0.667 \left(\frac{PL^3}{EI}\right)$, downward

Ex- 2.12



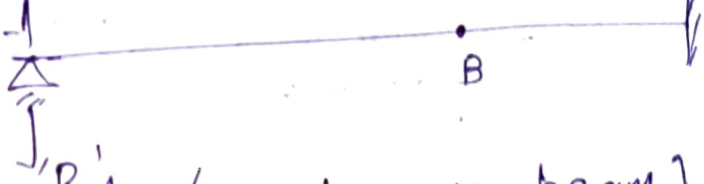
load of conjugate beam

$$\Sigma M_B = 0$$

$$R_A = 0$$

Total load of conjugate beam $= \frac{1}{3} \times \frac{2L}{3} \times \frac{wL^2}{18EI}$

$$= \frac{wL^3}{81EI}.$$



R_A' (conjugate beam)

C.G is at B,

$\Sigma M_B = 0$, conjugate beam

$$R_A' \times L = \int_0^{L/3} \frac{wx^2}{2EI}\left(\frac{L}{3} - x\right) dx$$

$$= \frac{w}{2EI}\left[\frac{L x^3}{3 \times 3} - \frac{x^4}{4}\right]_0^{L/3}$$

$$= \frac{wL^4}{2EI}\left(\frac{1}{9 \times 27} - \frac{1}{4 \times 81}\right)$$

$$\Rightarrow R_A' = \frac{wL^3}{1944EI}$$

At the free end C,

$$\theta_C = F_C = -R_A + \text{total load}$$

$$= \frac{wL^3}{1944EI} \mp \frac{1}{3} \times \frac{2L}{3} \times \frac{wL^2}{18EI}$$

$$\mp \frac{wL^3}{81EI}$$

$$= \frac{wl^3}{1944EI} + \frac{wl^3}{81EI} \cancel{+ \frac{wl^3}{}}$$

$$= \frac{wl^3}{1944EI}(-1+24)$$

$$= \frac{23}{1944EI} \times wl^3, \text{ clockwise}$$

Δ_C = Moment at 'C' in the conjugate beam

$$= -\frac{wl^3}{1944EI} \times \left(l - \frac{l}{3}\right) + \frac{wl^3}{81EI} \times \frac{l}{3}$$

$$= \frac{wl^4}{1944 \times 3EI}(-4+24)$$

$$= \frac{5}{1458EI} \times wl^4, \text{ downward}$$

Ex:- 2.13

Determine the rotation at A and defl^n at 'C' in the overhanging beam.

$$\Sigma M_B = 0$$

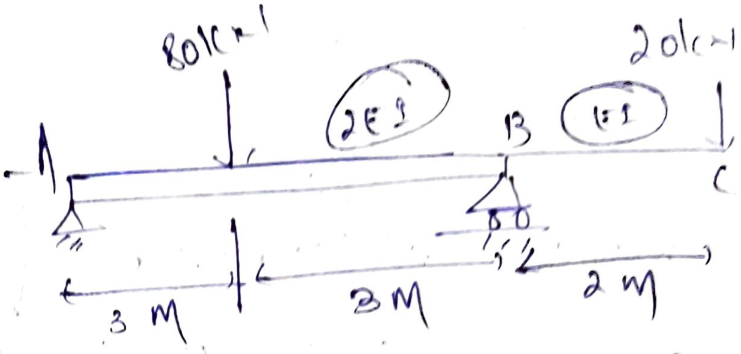
$$\Rightarrow R_A' \times 6 = \frac{1}{EI}\left(\frac{1}{2} \times \frac{20}{EI} \times 6 \times 2 + \frac{1}{2} \times \frac{60}{EI} \times 6 \times 3\right)$$

$$\Rightarrow R_A' = \frac{110}{EI}$$

$$\theta_A = \frac{110}{EI}, \text{ clockwise}$$

Defl^n at C :

= Moment at 'C' in conjugate beam



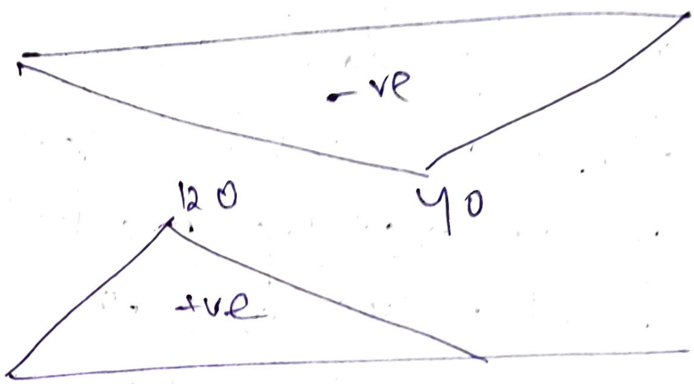
$$= \frac{110}{EI} \times 8 - \frac{1}{2} \times \frac{60}{EI} \times 6 \times 5$$

$$+ \frac{1}{2} \times 6 \times \frac{20}{EI} \times 4$$

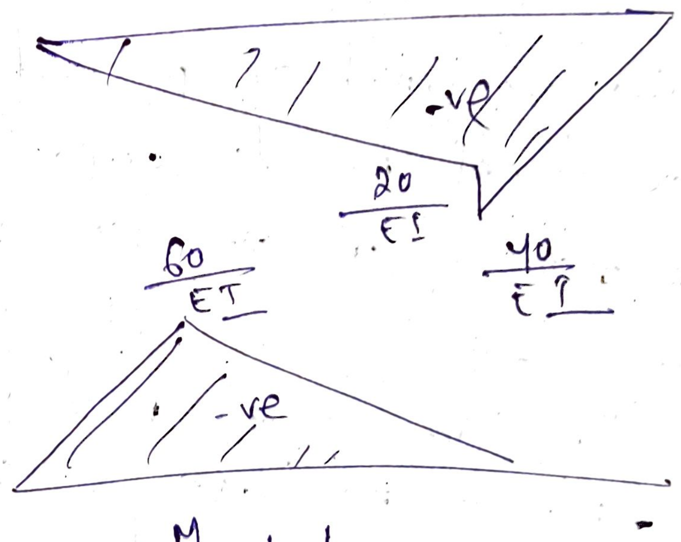
$$+ \frac{1}{2} \times \frac{40}{EI} \times 2 \times 1$$

$$= \frac{260}{EI}$$, downward

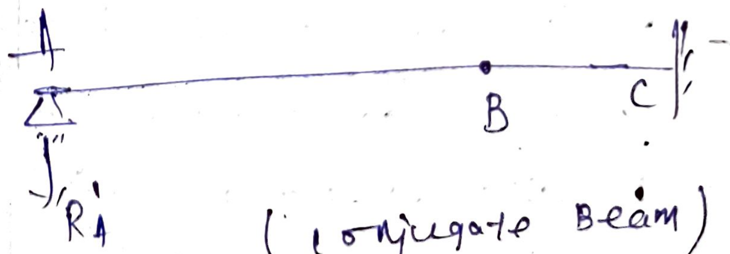
(Ans)



(BMD)



$\left(\frac{M}{EI}\right)$ dig.



(Conjugate Beam)

Strain Energy

Introduction:

→ when an external load acts on a structure, the str. undergoes deformation, hence the work is done.

→ to resist these external forces the internal forces develop gradually from zero to their final value and the work is done.

- this internal work done is stored as energy of the str. and its helps the str. to spring back to the original shape and size, whenever the external loads are removed, provided the material of the str. is still with in the elastic limit.

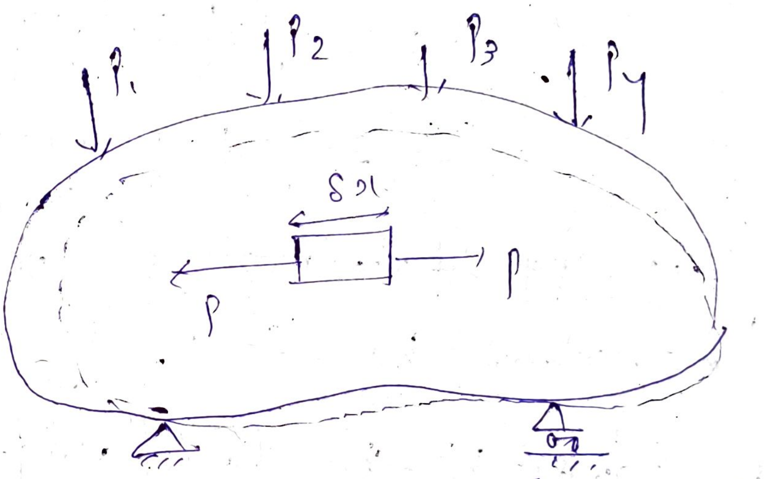
- this internal work, which is stored as energy is due to the straining of the material is called strain energy.

- when eqm is reached, as per the well known law of conservation of energy, the work done by the external forces must equal the strain energy stored.

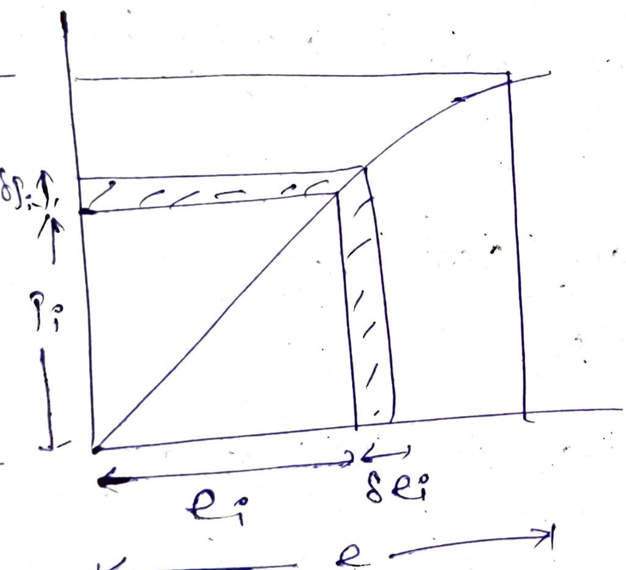
Methods for finding the defl. of beams :-

1. Strain energy / Real work method
(Complementary energy)
2. Virtual work / Unit load method
3. Castigliano's method.

Strain Energy



(Structural system)



(Stress- strain relation)

Stress - P_i
Strain - δe_i

Workdone = Force × Displacement

$= P_i \, \delta a \, \delta e_i \, \delta x$

$= P_i \, \delta e_i \, \delta v \quad (\because \delta v = \delta a . \delta x)$ vol.

Strain energy stored in the element $= \int_0^{e_i} p_i \, \delta e_i \, \delta v$

= Area under stress-strain curve $\times dv$

If stress-strain curve is linear,

Strain energy of the element $= \frac{1}{2} p e \, dv$

Strain energy stored in the str.,

$$U = \int \frac{1}{2} p e \, dv$$

$$= \int \frac{1}{2} \times \text{stress} \times \text{strain} \; dv$$

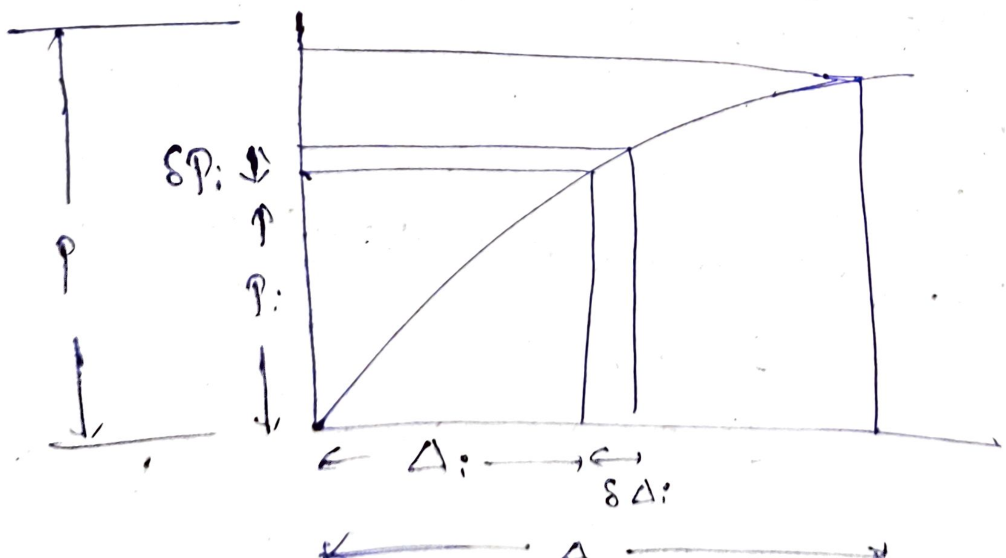
Let, during deformation the load acting be P_i and deformation be $\delta \Delta_i$.

Work done by the load under consideration

$$= \int P_i \, \delta \Delta_i$$

= Area under the load of deformation curve

$= \frac{1}{2} \times P \Delta \rightarrow$ linear elasticity prob.



(Load versus deformation curve)

workdone by external load,

$$\int_0^{1} \frac{1}{2} P\Delta = \Sigma \frac{1}{2} P\Delta$$

complementary energy of the element,

$$C.E = \int_0^{P_i} e_i \, \delta P_i \, dv$$

$$= \text{Area above the stress-strain curve} \times dv$$

$$= \int \frac{1}{2} pe \, dv$$ — linear elasticity prob.

C.E of the entire str., $U_c = \int \frac{1}{2} pe \, dv$ — "

complementary workdone $= \int_0^{P_i} \Delta_i \, \delta P_i$

workdone $= \frac{1}{2} \Delta P$ — linear elasticity prob.

if 'n' no. of loads, workdone by external load

$$= \Sigma \frac{1}{2} \Delta P$$

Strain energy due to Bending :-

(A beam under pure moment)

$$\boxed{f = \left(\frac{M}{I}\right) y}$$

where, M = Bending Moment

I = MOI of the section

Strain, $e = \frac{f}{E}$

$$= \left(\frac{M}{EI}\right) y$$

Where,

E = Young's Modulus.

SE in the element $= \frac{1}{2} \times \text{stress} \times \text{strain} \times dv$

$$= \frac{1}{2} \times \left(\frac{M}{I}\right)y \times \left(\frac{M}{EI}\right) y \times dv$$

$$= \frac{1}{2} \times \frac{M^2 y^2}{EI^2} \times dv$$

SE in the beam $= \int \frac{1}{2} \times \frac{M^2 y^2}{EI^2} \times dv$

$$= \int_0^L \int_0^A \frac{1}{2} \times \frac{M^2 y^2}{EI^2} \, \delta a \, dx$$

$\because A$ = cross sectional area

$$= \int_0^L \frac{M^2}{2EI^2} \left(\int_0^A y^2 \, \delta a \right) dx$$

$$= \int_0^L \frac{M^2}{2EI^2} I \, dx \qquad \because \int_0^A y^2 \, \delta a = I$$

$$\Rightarrow \boxed{U = \int_0^L \frac{M^2}{2EI} dx}$$

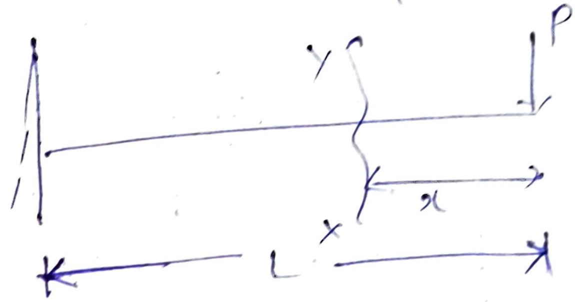
From the law of conservation of energy,

Strain energy = Real work done by loads

$$\rightarrow \boxed{U = \sum_0^n \frac{1}{2} P \Delta}$$

Ex. - 3.1

Using SE method determine the defl^n of the free end of a cantilever of length 'L' subjected to a concentrated load 'p' at the free end.



Ans :-

B.M at a distance 'x' from the free end :-

$$M = px$$

$$S.E = \int_0^L \frac{M^2}{2EI}\,dx$$

$\because \Delta$ = defl^n at the free end

$$= \int_0^L \frac{p^2 x^2}{2EI}\,dx$$

$$= \frac{p^2}{2EI}\left[\frac{x^3}{3}\right]_0^L$$

$$= \frac{p^2}{2EI} \times \frac{1}{3}(L^3 - 0)$$

$$= \frac{p^2 L^3}{6EI}$$

workdone by the load $= \frac{1}{2} P \Delta$

S.E = work done by external loads

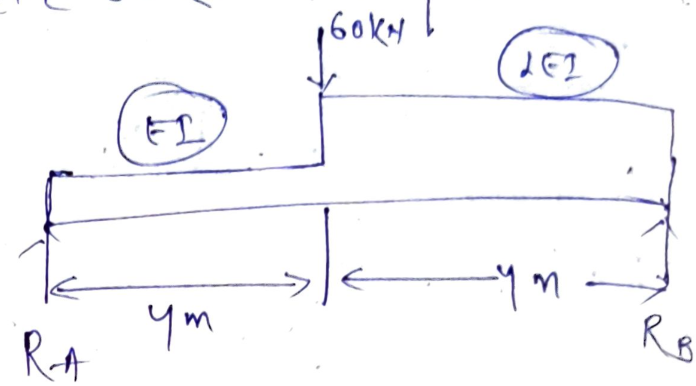
$$\Rightarrow \frac{p^2 L^3}{6EI} = \frac{1}{2} P \Delta$$

$$\Rightarrow \Delta = \frac{PL^3}{3EI}$$

Ex:- 3.2

Determine the defl in center 60kN load in the beam

Ans:-



$R_A = R_B = 30\,kN$

B.M at any distance 'x' from 'A' or at a distance 'x' from B.

$$S.E = SE_1 + SE_2$$

$$\Rightarrow U = \int_0^4 \frac{(30x)^2}{2EI}\,dx + \int_0^4 \frac{(30x)^2}{2\times 2EI}\,dx$$

$$\Rightarrow U = \int_0^4 \frac{900x^2}{2EI}\,dx + \int_0^4 \frac{900x^2}{4EI}\,dx$$

$$\Rightarrow U = \frac{900}{2EI}\int_0^4 x^2\,dx + \frac{900}{4EI}\int_0^4 x^2\,dx$$

$$= \frac{900}{2EI}\left[\frac{x^3}{3}\right]_0^4 + \frac{900}{4EI}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{900}{2EI}\times\frac{1}{3}\times(4)^3 + \frac{900}{4EI}\times\frac{1}{3}\times(4)^3$$

$$= \frac{900}{2EI}\times\frac{1}{3}\times(4)^3\left(1+\frac{1}{2}\right)$$

$$= \frac{900}{\cancel{6}EI}\times\cancel{64}\times\frac{\cancel{3}}{\cancel{2}}$$

$$= \frac{14400}{EI}$$

work done by the load $= \frac{1}{2} \times P \Delta$

$= \frac{1}{2} \times 60 \times \Delta$

Equating S.E of the beam to the work done by load,

$$\frac{14400}{EI} = \frac{1}{2} \times 60 \times \Delta$$

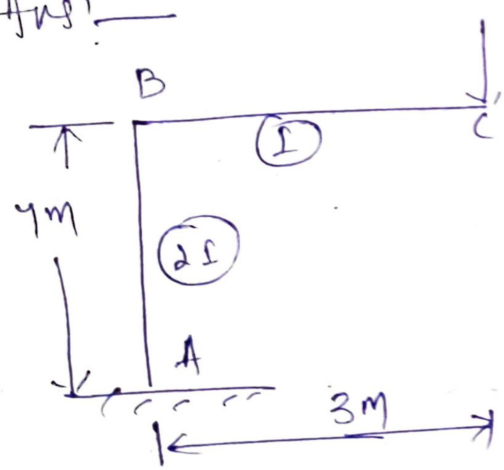
$$\Rightarrow \Delta = \frac{14400 \times 2}{60 \times EI}$$

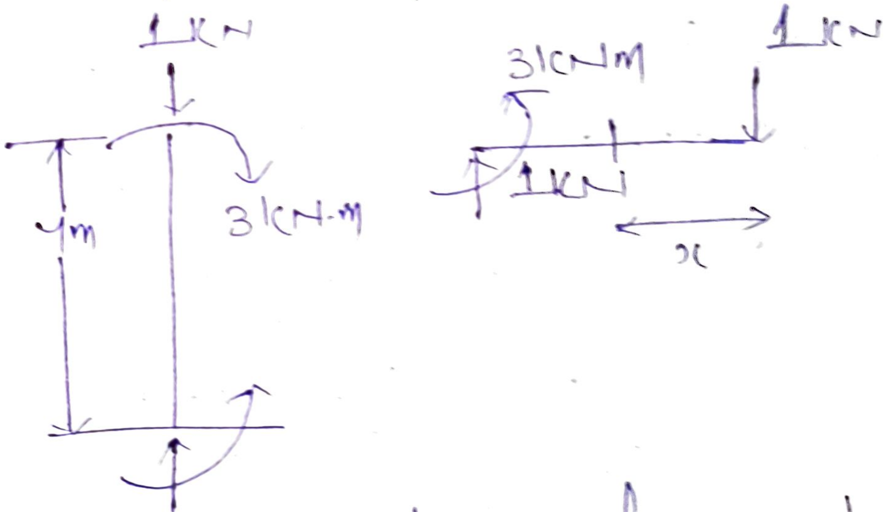
$$\Rightarrow \Delta = \frac{480}{EI}$$

Ex:- 3.3

Determine the vertical defl^n of pt. 'C' in the frame. Give $E = 200\ kN/mm^2$ and $I = 30 \times 10^6\ mm^4$.

Ans:-





(FBD of the frame)

Portion	Origin	Limit	Expression
BC	C	0-3	$1.x = x$
AB	B	0-4	3

$SE = SE_1 + SE_2$

$$= \int_0^3 \frac{(x)^2}{2EI} dx + \int_0^4 \frac{(3)^2}{2E \times 2I} dx$$

$$= \frac{1}{2EI}\int_0^3 x^2\,dx + \frac{9}{4EI}\int_0^4 dx$$

$$= \frac{1}{2EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{9}{4EI}\left[x\right]_0^4$$

$$= \frac{1}{2EI}\times\frac{1}{3}\times(3)^3 + \frac{9}{\cancel{4}EI}\times\cancel{4}$$

$$= \frac{9}{2EI} + \frac{9}{EI}$$

$$= \frac{9}{EI}\left(\frac{1}{2}+1\right)$$

$$= \frac{9}{EI}\times\frac{3}{2}$$

$$= \frac{27}{2EI}$$

work done $= \frac{1}{2}\times 1\times\Delta = \frac{\Delta}{2}$

So $\frac{\Delta}{\cancel{2}} = \frac{27}{\cancel{2}EI}$

$\Rightarrow \Delta = \frac{27}{EI}$ ~~(Ans)~~

$$= \frac{27}{200\times10^{-6}\times 30\times10^{6}}$$

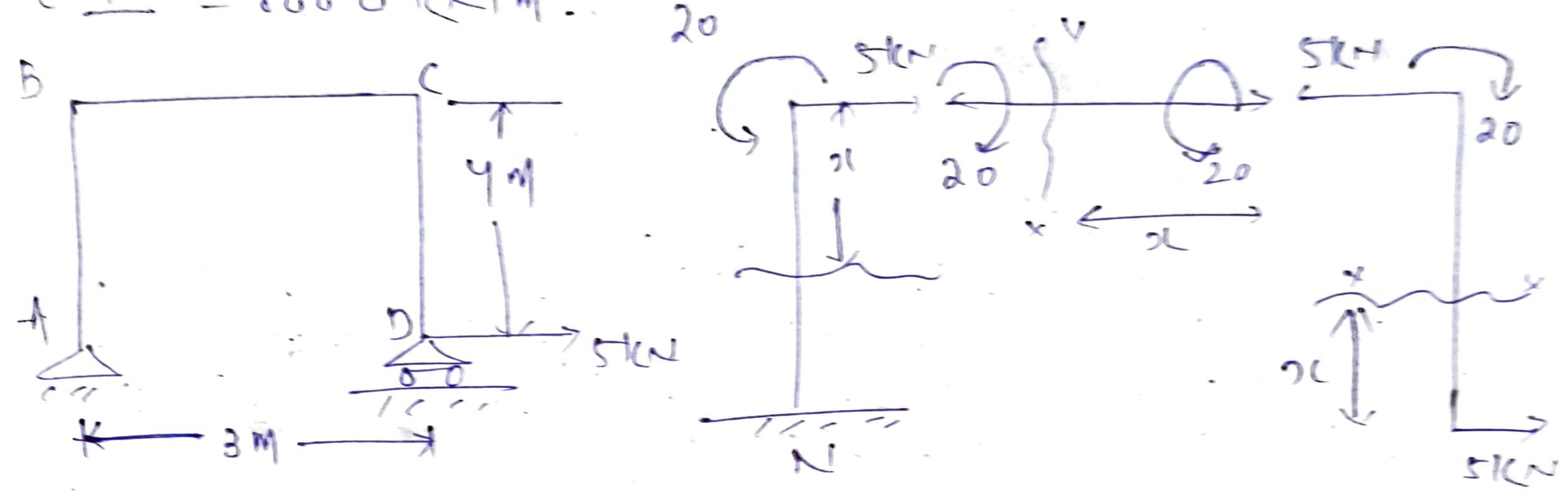
$= 0.045\,m$

$= 4.5\,mm$

Ex-3.4

Determine the horizontal displacement of the roller end 'D' of the portal frame.

$EI = 8000\ kNm^2$.



Portion	CD	BC	AB
origin	D	C	B
Limit	0-4	0-3	0-4
M_x	$5x$	20	$20-5x$

$SE = SE_1 + SE_2 + SE_2$

$$= \int_0^4 \frac{(5x)^2}{2EI}dx + \int_0^3 \frac{(20)^2}{2EI}dx + \int_0^4 \frac{(20-5x)^2}{2EI}dx$$

$$= \frac{25}{2EI}\int_0^4 x^2 dx + \frac{400}{2EI}\int_0^3 dx + \frac{1}{2EI}\int_0 (400 - 200x + x^2)\,dx$$

$$= \frac{25}{2EI}\left[\frac{x^3}{3}\right]_0^4 + \frac{400}{2EI}[x]_0^3 +$$

$$\frac{1}{2EI}\left\{400[x]_0^4 - 200\left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x^3}{3}\right]_0^4\right\}$$

$$= \frac{25}{2EI} \times \frac{1}{3} \times (4)^3$$

$$+ \frac{400}{2EI} \times 3$$

$$+ \frac{1}{2EI} \times 400 \times 4$$

$$- \frac{1}{2EI} \times 200 \times \frac{1}{2} \times (4)^2$$

$$+ \frac{1}{2EI} \times \frac{1}{3} \times (4)^3$$

$$= \frac{25 \times \cancel{64}^{32}}{\cancel{2} \times 3 \times EI} + \frac{\cancel{400}^{200} \times 3}{\cancel{2} EI} + \frac{400 \times \cancel{4}^{2}}{\cancel{2} EI} - \frac{200 \times \cancel{16}^{4}}{\cancel{2} EI \times \cancel{2}} + \frac{\cancel{64}^{32}}{\cancel{2} EI \times 3}$$

$$= \frac{25 \times 32}{3EI} + \frac{32}{3EI} + \frac{600}{EI} + \cancel{\frac{800}{EI}} - \cancel{\frac{800}{EI}}$$

$$= \frac{32}{3EI}(25+1) + \frac{600}{EI}$$

$$= \frac{32 \times 26}{3EI} + \frac{600}{EI}$$

$$= \frac{1133.33}{EI}$$

work done $= \frac{1}{2} \times P \times \Delta = \frac{1}{2} \times 5 \times \Delta = 2.5\,\Delta$

So $2.5\,\Delta = \frac{1133.33}{EI}$

$$\Rightarrow \Delta = \frac{1133.33}{EI \times 2.5}$$

$$= \frac{453.33}{8000}$$

$$= 0.05667 \text{ m}$$

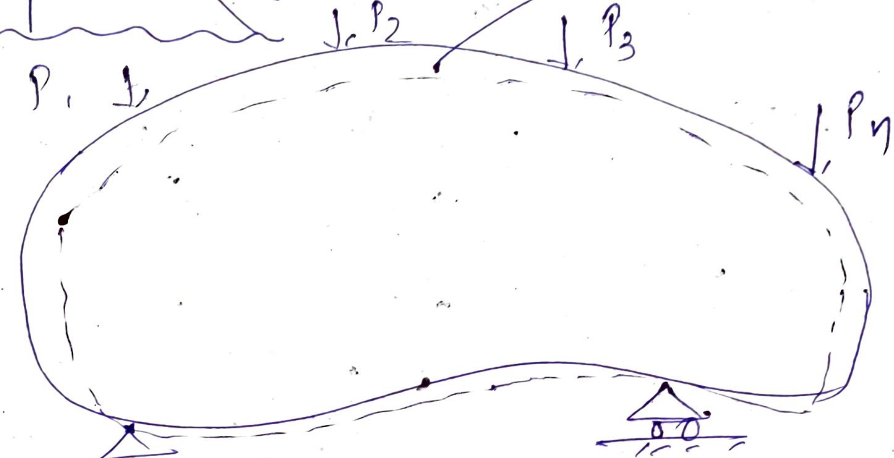
$$= 56.7 \text{ mm} \quad (\text{Ans})$$

NOTE

1. the str. is s.t. a single conc. load.
2. Defl required is at the loaded pt. and is in the dirn of the load.

virtual work :-

hypothetical deformed shape.



(A body s.t. real loads and virtual displacement)

Various results have shown that the SE/real work method is having limited use in finding displacement.

- The method based on virtual work principle proves to be the most versatile and powerful method.

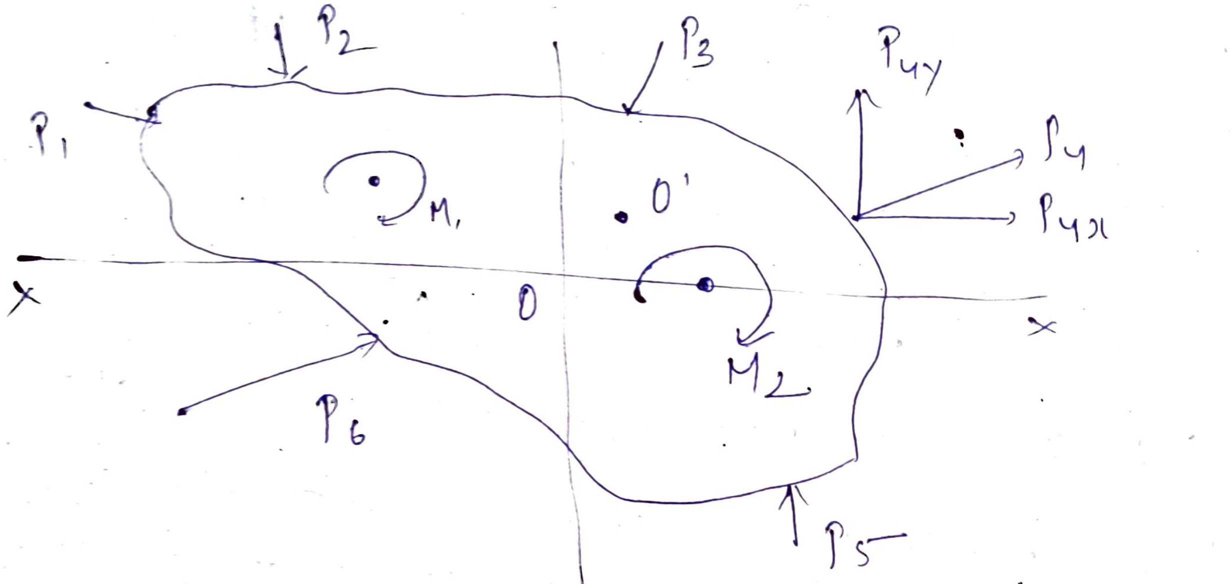
P_1, P_2, P_3 - - - P_n → real forces..

- The hypothetical deformation is called virtual deformation and the workdone by real forces due to virtual displacement is called virtual work.
- Virtual means the effect exists but actually it is not the fact.

- the virtual work may be defined as the work done by real forces due to hypothetical displacements or the work done by hypothetical forces during real displacement.

1717 (Johann Bernoulli)

Bernoulli's principle of virtual Displacement



(A body in eq^n s.t. loads & external moments)

consider the rigid body, s.t. 'p' system of forces and 'M' system of moments.

Let P_x, P_y → components of force in x-dir^n and y-dir^n

the body is in eq^m.

$$\Sigma P_x = 0$$

$$\Sigma P_y = 0$$

$$\Sigma M + \Sigma P_x y + \Sigma P_y x = 0$$

OO' → virtual displacement

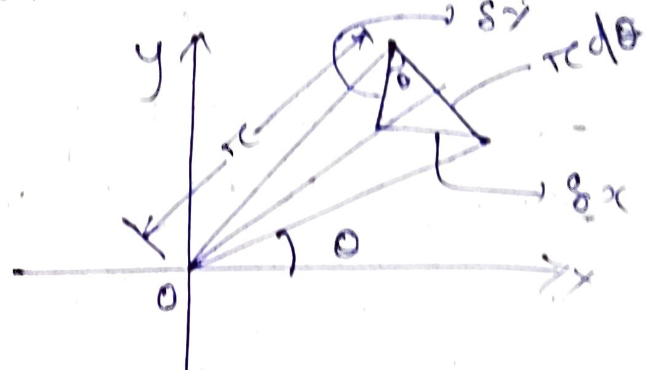
Let the component of OO' in x direc^n be δx and in y direc^n be δy.

Virtual work done : $\Sigma P_x \delta x + \Sigma P_y \delta y$

$$= \delta x \, \Sigma P_x + \delta y \, \Sigma P_y = 0$$

Now consider the rotation of the rigid body by a virtual rotation '$d\theta$':

If a pt. $Q(x, y)$ is at a distance 'r' from the origin.



$$\delta x = r\,d\theta \sin\theta$$
$$= y\,d\theta$$

$$\delta y = r\,d\theta \cos\theta$$
$$= x\,d\theta$$

(An element of rigid body s.t. virtual rotation)

Virtual workdone by real forces due to virtual displacements

$$= \Sigma M\,d\theta + \Sigma P_x \delta x + \Sigma P_y \delta y$$

$$= \Sigma M\,d\theta + \Sigma P_x y\,d\theta + \Sigma P_y x\,d\theta$$

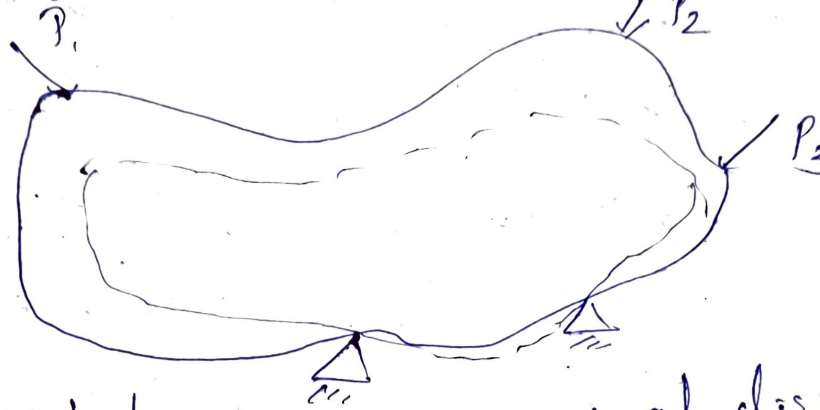
$$= d\theta\,(\Sigma M + \Sigma P_x y + \Sigma P_y x)$$

So $\Sigma M + \Sigma P_x y + \Sigma P_y x = 0$

⇒ virtual work = 0

* If a rigid body is in eq^m under a system of forces and/or moments, the virtual work done by this system of forces and/or moments during virtual displacement is zero.

Principle of virtual work for Deformable Bodies



(An elastic body s.t. forces virtual displacements)

Consider an elastic body s.t. a system of forces.

- These forces causes real internal stresses and every element in the body is in eqm. under the action of external forces and internal stresses.
- Imagine a virtual displacement which means displacement due to some hypothetical force system.
- Let dw_e be virtual work of the external forces acting on an element.
- Since the body is elastic, there will be virtual displacement and also virtual deformation (strain).
- The external work applied to the element is dissipated in two forms. i.e :-

1. the virtual work, dw_r, treating the element as a rigid body.
2. the virtual work of deformation of the element dw_i. (virtual strain energy of the element).

$$dw_\theta = dw_i + dw_{re}$$

From the principle of virtual displacement,

$$dw_{re} = 0$$

$$dw_\theta = dw_i$$

Integrating both sides:-

$$\boxed{W_\theta = W_i}$$

where,

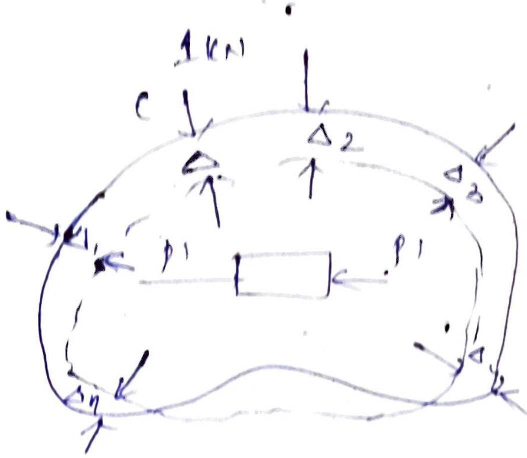
W_θ → total virtual workdone by force system

W_i → Internal virtual S.E of the entire body.

* If a deformable body in eqm under a system of forces is given virtual deformation, the virtual workdone by the system of forces is equal to the internal virtual workdone by the stresses due to that system of forces.

Unit Load Method :—

(A body s.t. load and deformed shape)



(System of forces applied to a body)

- Consider the body which is s.t. forces $P_1, P_2, P_3, P_4 \ldots P_n$ applied gradually.
- Let displacement under load at points by $\Delta_1, \Delta_2, \Delta_3 \ldots \Delta_n$
- at pt. C be Δ.

External work done $= \frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \frac{1}{2}\Delta_3 P_3 + \ldots + \frac{1}{2}\Delta_n P_n$

S.E stored $= \int \frac{1}{2} pe\, dv$

where,

p = stress

e = strain

$$\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \ldots + \frac{1}{2}\Delta_n P_n = \int \frac{1}{2} pe\, dv$$

Consider the same body s.t. an unit load applied gradually at 'C' when it is removed system of P forces.

Let the displacement at 1, 2, 3 .... n be $\delta_1, \delta_2, \delta_3 ... \delta_n$ respectively and the displacement at 'C' be δ.
Let the stress produced in the element be p' and the strain be e'.

External workdone $= \frac{1}{2} \times 1 \times \delta$

Internal workdone $= \int \frac{1}{2} p'e' dv$

$$\boxed{\frac{1}{2} \times 1 \times \delta = \int \frac{1}{2} p'e' dv}$$

If 'p' system of forces is applied to the body.

External workdone $= \frac{1}{2} \Delta_1 P_1 + \frac{1}{2} \Delta_2 P_2 + \text{———} + \frac{1}{2} \Delta_n P_n + 1 \times \Delta$

Internal workdone $= \int \frac{1}{2} pe\, dv + \int p e\, dv$

Since the stress p' is acting throughout the deformation

Equating internal work to external work,

$$\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \text{———} + \frac{1}{2}\Delta_n P_n + 1 \times \Delta = \int \frac{1}{2} pe\, dv + \int p'e\, dv$$

$$\Rightarrow 1 \times \Delta = \int p'e\, dv$$

$$\Rightarrow \Delta = \int p'e\, dv$$

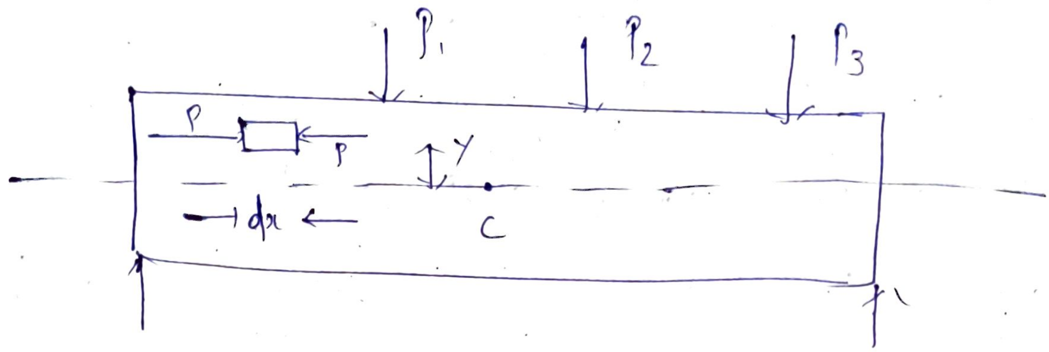
Where,

Δ = defl^n at pt. where unit load is applied and is measured in the direct^n of unit load

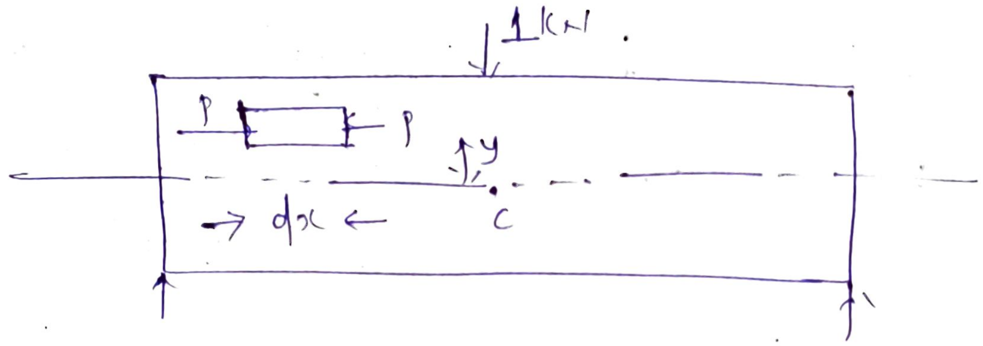
p' = stress in an element due to unit load

e' = strain in an " " given load system.

The unit load method - Application to beam deflections :→



(Beam s.t. P forces)



(Beam s.t. to unit load)

Stress in the element at distance 'y' from N.A is

$$\boxed{p = \frac{M}{I} y'}$$

where, M = Moment acting at the section.

Strain in the element due to given system of forces, $e = \frac{M}{EI} y$

Stress, $p' = \frac{My}{I}$

$$\Delta = \int \frac{M}{I} y + \frac{M}{EI} y \, dv$$

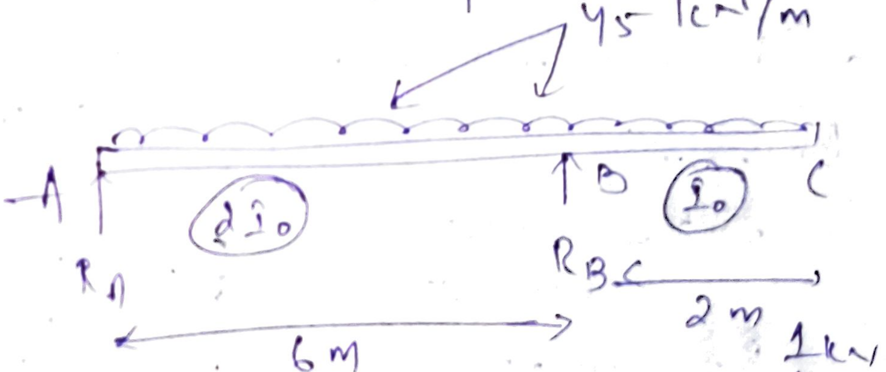
$$\Delta = \int_0^L \frac{Mm}{EI^2}\left(\int_0^A y^2 \, dA\right) dx$$

$$= \int_0^L \frac{Mm}{EI^2} I \, dx \qquad \left(\because \int_0^A y^2 \, dA = I\right)$$

$$= \int_0^L \frac{Mm}{EI} dx$$

Ex:- 3.5

Determine the defl^n at free end of the overhanging beam. Use unit load method.



Ans:-

$\Sigma M_A = 0$

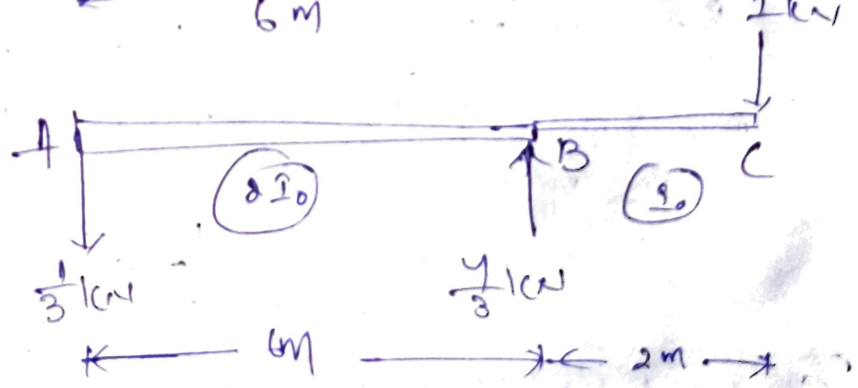
$\Rightarrow R_B \times 6 = 45 \times 8 \times 4$

$\Rightarrow R_B = 240 \, kN$

$\Sigma V = 0$

$\Rightarrow R_A = 45 \times 8 - 240$

$= 120 \, kN$



(FBD of beam w.r.t unit load)

When unit load is acting at C,

$R_B = \frac{1 \times 8}{6} = 1.333 \, kN$

$R_A = 0.333 \, kN \, (\downarrow)$

Portion	AB	BC
Origin	A	C
Limit	0 – 6	0 – 2
M	$120x - \frac{1}{2} \times 45x^2$	$-\frac{1}{2} \times 45x^2$
m	$-0.333x$	$-x$
I	$2I_0$	I_0

Ex. 3.7

Determine the defln at the free end of the overhanging beam. Assume uniform flexural rigidity.

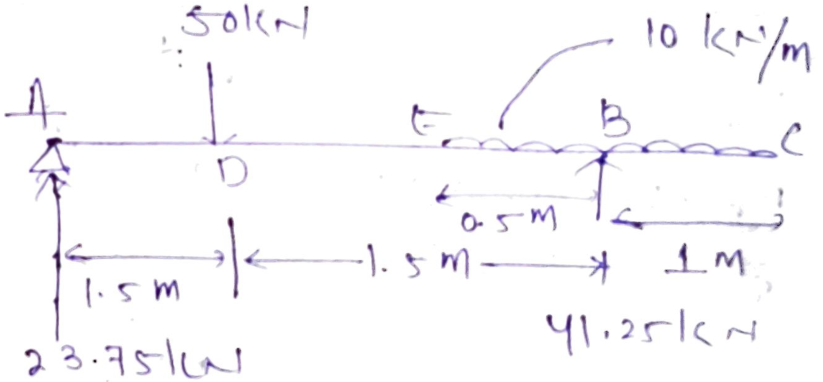
Ans:–

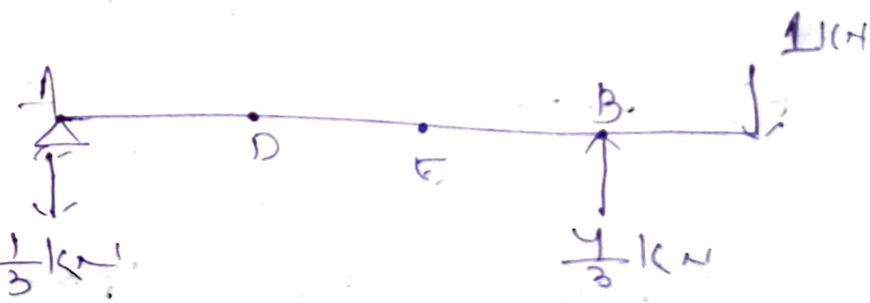
$R_B \times 3 = 50 \times 1.5 + 10 \times 1.5 \times 3.25$

$\Rightarrow R_B = 41.25$

$R_A = 50 + 1.5 \times 10 - 41.25$

$= 23.75$ kN





(Beam with unit vertical load at C)

Sagging Moment → +ve
Hogging Moment → −ve

portion	AD	DE	EB	BC
Origin	A	D	B	C
Limit	0 - 1.5	0 - 1	0 - 0.5	0 - 1
M	23.75 m	$23.75(x+1.5) - 50x$	$41.25x - \frac{10(x+1)^2}{2}$	$-\frac{10x^2}{2}$
m	$-\frac{1}{3}x$	$-\frac{1}{3}(x+1.5)$	$-(1+x)+\frac{4}{3}x$	$-1.x$
Flexural rigidity	EI	EI	EI	EI

$$\Delta_C = \int_0^{1.5} \frac{23.75x}{EI} \times \frac{(-x)}{3}\,dx$$

$$+ \int_0^1 \frac{[23.75(x+1.5)-50x][-(1/3)(x+1.5)]}{EI}\,dx$$

$$+ \int_0^{0.5} [41.25x - 5(x+1)^2]\left[-1+\frac{x}{3}\right]dx$$

$$+ \int_0^1 \frac{5x^3}{EI}\,dx$$

$$= \int_0^{1.5}\left[-\frac{23.75}{9} \times \frac{x^2}{EI}\right]dx$$

$$-\frac{1}{3}\int_0^1 \frac{[-26.25x^2 - 3.75x + 53.438]}{EI}\,dx$$

$$+ \int_0^{0.5} \frac{\left[-\frac{5}{3}x^3 - 15.417x^2 - 32.917x + 5\right]}{EI}\,dx$$

$$+ \int_0^1 \frac{5x^3}{EI}\,dx$$

$$= \frac{1}{EI}\left\{\left[-\frac{23.75x^3}{9}\right]_0^{1.5} + \frac{1}{3}\left[\frac{26.25x^3}{3} + 3.75\times\frac{x^2}{2} - 53.438x\right]_0^1 \right.$$

$$\left. + \left[-\frac{5x^4}{12} + 15.417\times\frac{x^3}{3} - 32.917\times\frac{x^2}{2} + 5x\right]_0^{0.5} + \left[\frac{5x^4}{4}\right]_0^1\right\}$$

$$\Rightarrow \Delta_C = \frac{1}{EI}(-8.906 - 14.231 - 0.798 + 1.25)$$

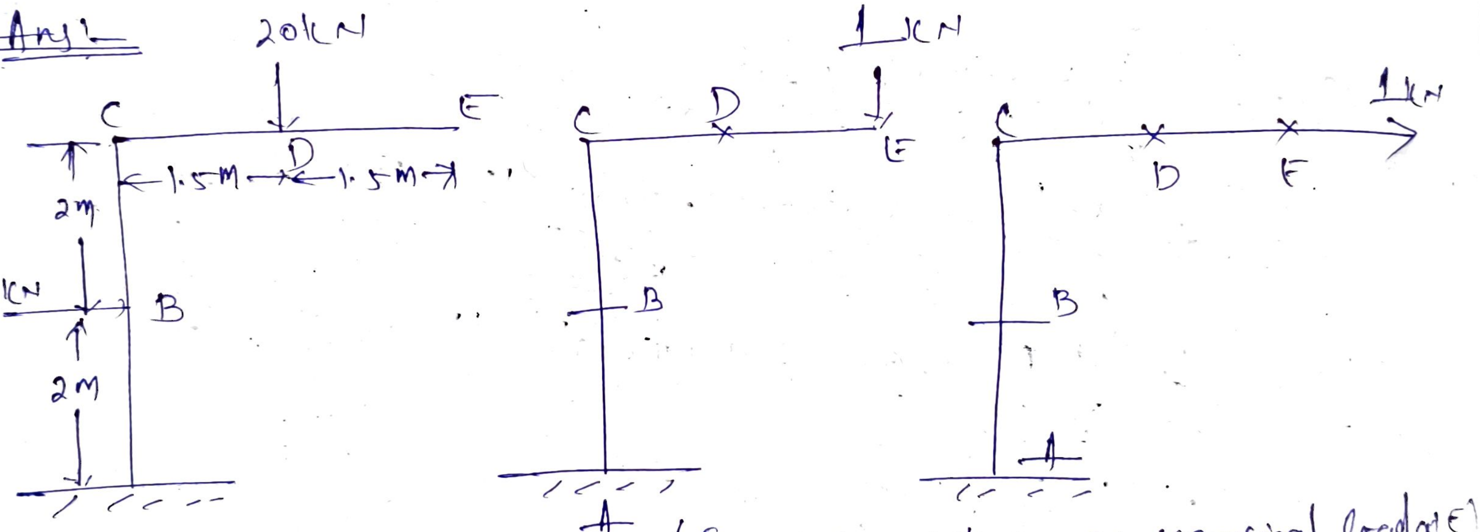
$$= \frac{-22.885}{EI}$$

$$= \frac{22.885}{EI} \text{ (Upward)}$$

Ex:- 3.8

Determine the vertical and horizontal deflection at the free end of the bent. Assume uniform flexural rigidity EI throughout.

Ans:-



(Frame with unit vertical load at E)

M → Moments due to given loads

m_1 → " " " unit vertical load at the free end

m_2 → " " " " Horizontal " " " "

Portion	ED	DC	CB	BA
Origin	E	D	C	B
Limit	0 - 1.5	0 - 1.5	0 - 2	0 - 2
M	0	$-20.x$	-30	$-30-10x$
m_1	x	$-(1.5+x)$	-3	-3
m_2	0	0	$-x$	$-(x+2)$
Flexural Rigidity	EI	EI	EI	EI

$$EI\,\Delta_{EV} = \int M m_1\,dx$$

$$= 0 + \int_0^{1.5} 20x(1.5+x)\,dx + \int_0^{2} 90\,dx + \int_0^{2} (90+30x)\,dx$$

$$= \int_0^{1.5} (30x+20x^2)\,dx + \int_0^{2} 90\,dx + \int_0^{2} (90+30x)\,dx$$

$$= \left[\frac{30x^2}{2} + \frac{20x^3}{3}\right]_0^{1.5} + \left[90x\right]_0^{2} + \left[90x + \frac{30x^2}{2}\right]_0^{2}$$

$$= 56.25 + 180 + 240$$

$$= 476.25$$

$$EI\,\Delta_{EH} = \int M m_2\,dx$$

$$= 0 + 0 + \int_0^{2} 30x\,dx + \int_0^{2} (30+10x)(x+2)\,dx$$

$$= \left[15x^2\right]_0^{2} + \int_0^{2} (10x^2 + 50x + 60)\,dx$$

$$= 60 + \left[\frac{10x^3}{3} + 50 \times \frac{x^2}{2} + 60x\right]_0^{2}$$

$$= 306.67$$

$$\Delta_{EV} = \frac{476.25}{EI}$$

$$\Delta_{EH} = \frac{306.67}{EI}$$

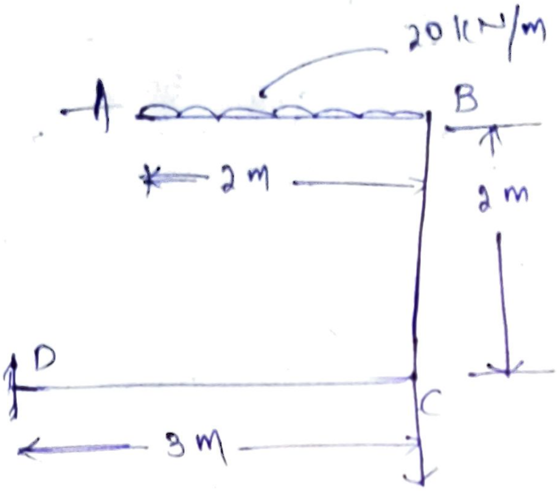
Ex. 3.9

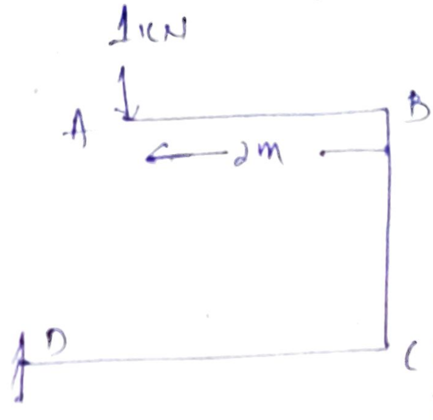
Determine the vertical defln at A and C in the frame.

Take,

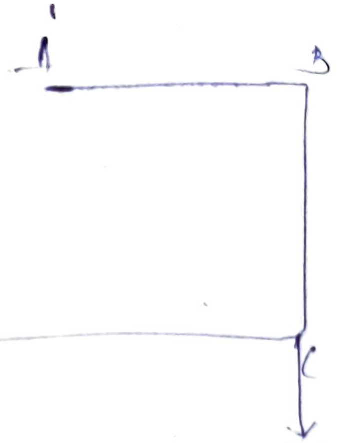
$E = 200$ GPa (Moment -ve compression)

$I = 150 \times 10^4$ mm^4





Frame with unit vertical load at A



Frame with unit moment at D

Portion	AB	BC	CD
Origin	A	B	C
Limit	0-2	0-2	0-3
M	$10x^2$	40	$40-130x$
m_1	x	2	$2-x$
m_2	0	0	$-x$
Flexural rigidity	EI	EI	EI

$$EI\,\Delta_A = \int_0^2 10x^2 \cdot x\,dx + \int_0^2 80\,dx + \int_0^3 (40-130x)(2-x)\,dx$$

$$= \left[\frac{10x^4}{4}\right]_0^2 + \left[80x\right]_0^2 + \int_0^3 (80-300x+130x^2)\,dx$$

$$= \frac{10(2)^4}{4} + (80\times2) + \left[80x - 300\frac{x^2}{2} + \frac{130x^3}{3}\right]_0^3$$

$$= \frac{10\times16}{4} + 160 + \left[80\times3 - 150(3)^2 + \frac{130}{3}(3)^3\right]$$

$$= 260$$

$E = 240\ GPa = 240\times10^9\ N/m^2$

$I = 150\times10^4\ mm^4 = 150\times10^4\times10^{-12}\ m^4 = 150\times10^{-8}\ m^4$

$$\Delta_A = \frac{260}{EI}$$

$$= \frac{260}{240\times10^9\times150\times10^{-8}}$$

$$= 7.222\times10^{-4}\ m$$

$$EI\,\Delta_C = \int M m_2\,dx$$

$$= 0 + 0 + \int_0^3 (40-130x)(-x)\,dx$$

$$= \int_0^3 (-40x + 130x^2)\,dx$$

$$= \left[-20x^2 + 130\frac{x^3}{3}\right]_0^3$$

$$= -20(3)^2 + \frac{130}{3}\times(3)^3$$

$$= 990$$

$$\Delta_C = \frac{990}{EI}$$

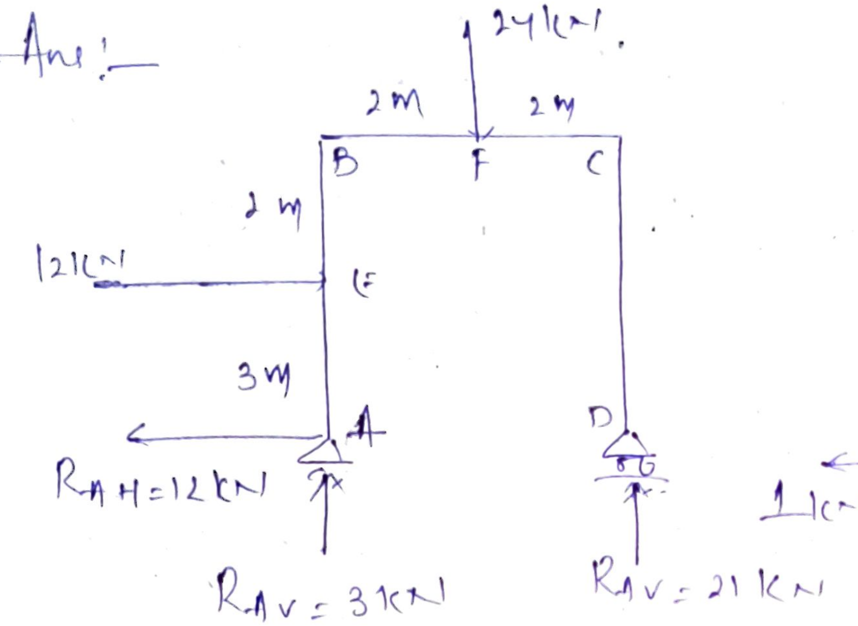
$$= \frac{990}{240\times10^9\times150\times10^{-8}} = 2.75\times10^{-3}\ m = 2.75\ mm$$

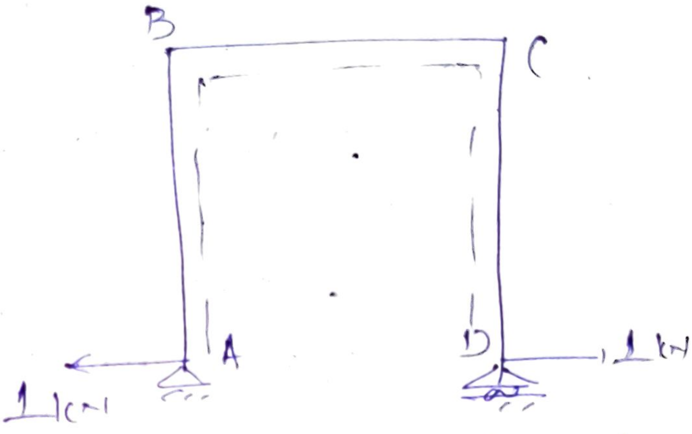
(ans)

Ex:- 3.10

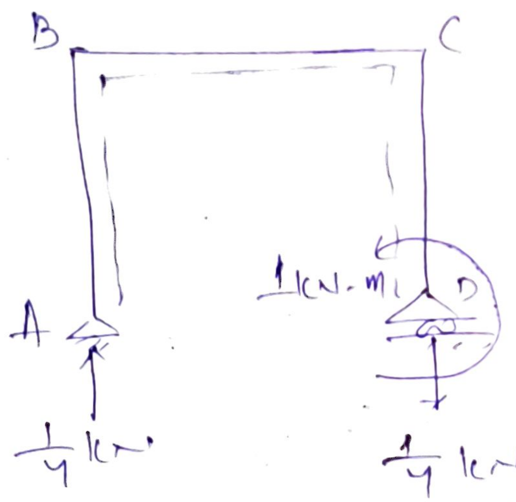
Determine the horizontal displacement and rotation at roller support in the frame. Flexural rigidity EI is const. throughout.

Ans:-





Frame with unit Horizontal load at D



(Frame with unit Moment at D)

$R_{DV} \times 4 = 24 \times 2 + 12 \times 3$

$\Rightarrow R_{DV} = 21$ kN

$R_{AV} = 24 - 21$

$= 3$ kN

$R_{AH} = 12$ kN

* Moment causing tension on dotted side is +ve

For unit Horizontal load at D (m_1 values)

for " Moment at D (m_2 values)

- All moment expressions are evaluated here taking forces from the end D.

Portion	AE	EB	BF	FC	CD
Origin	E	B	F	C	D
Limit	0 - 3	0 - 2	0 - 2	0 - 2	0 - 5
M	$36-12x$	$21\times4 - 24\times2 = 36$	$21(x+2) - 24x = 42-3x$	$21x$	0
m_1	$3-x$	$5-x$	5	5	x
m_2	0	0	$1-\frac{1}{4}(x+2) = \frac{1}{2}-\frac{x}{4}$	$1-\frac{x}{4}$	1
Flexural rigidity	EI	EI	EI	EI	EI

$$EI\,\Delta_{DH} = \int M m_1\, dx$$

$$= \int_0^3 (36-12x)(3-x)\,dx + \int_0^2 36(5-x)\,dx + \int_0^2 (42-3x)\,5\,dx + \int_0^2 21x(5)\,dx + 0$$

$$= \int_0^3 (108-72x+12x^2)\,dx + \int_0^2 (180-36x)\,dx + \int_0^2 (210-15x)\,dx + \int_0^2 105x\,dx$$

$$= \left[108x - 36x^2 + 4x^3\right]_0^3 + \left[180x - 18x^2\right]_0^2 + \left[210x - 7.5x^2\right]_0^2 + \left[52.5x^2\right]_0^2 = 996$$

$$\Rightarrow \Delta_{DH} = \frac{966}{EI}$$

$$EI\,\theta_D = \int M m_2\, dx$$

$$= 0 + 0 + \int_0^2 (42-3x)\left(\frac{1}{2}-\frac{x}{4}\right)dx + \int_0^2 21x\left(1-\frac{x}{4}\right)dx + 0$$

$$= \int_0^2 \left(21-12x+\frac{3x^2}{4}\right)dx + \int_0^2 (21x - 5.25x^2)\,dx$$

$$= \left[21x - 6x^2 + \frac{x^3}{4}\right]_0^2 + \left[\frac{21x^2}{2} - \frac{5.25x^3}{3}\right]_0^2 = 48$$

$$\theta_D = \frac{48}{EI}$$

Castigliano's Theorems :— (1879)

?

1st thm
(to determine defl^n)

2nd thm
(use in determining redundant reaction component)

In a linearly elastic structure, partial derivative of the S.E with respect to a load is equal to the defl^n of the pt. where the load is acting, the defl^n being measured in the dir^n of the load.

- the load may be force or a moment.

Mathematically,

$$\frac{dU}{dP_i} = \Delta_i$$

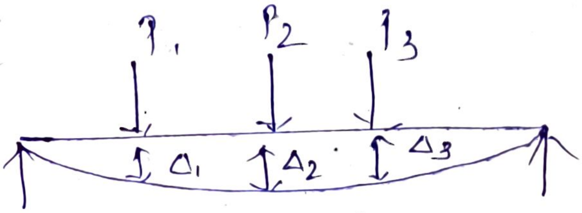
$$\frac{dU}{dM_j} = \theta_j$$

Where,

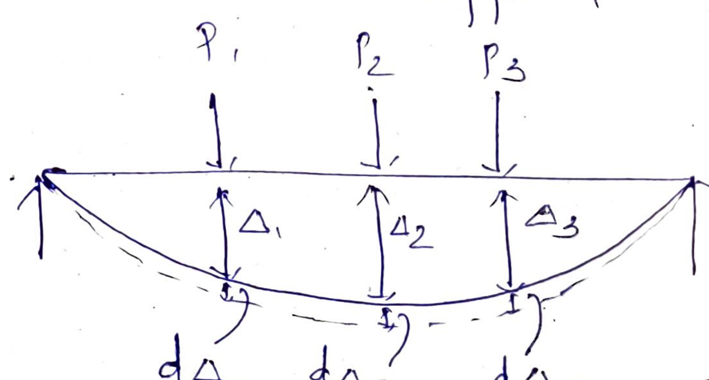
U = total S.E

$P_i, M_j \rightarrow$ loads

$\Delta_i, \theta_j \rightarrow$ defl^n



(SSB with gradually applied loads)



(SSB s.t to additional load)

Consider a SSB on which loads P_1, P_2 and P_3 are applied gradually.

Let the defl^n under the loads P_1, P_2 and P_3 be Δ_1, Δ_2 & Δ_3 respectively.

$$U = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3$$

let the additional load dP_1 be added after the loads P_1, P_2 and P_3 are applied.

let the additional defl^n be $d\Delta_1$, $d\Delta_2$, $d\Delta_3$.

the additional SE stored = dU

$$dU = \frac{1}{2}dP_1\,d\Delta_1 + P_1\,d\Delta_1 + P_2\,d\Delta_2 + P_3\,d\Delta_3$$

Total SE of the system,

$$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3$$
$$+ \frac{1}{2}dP_1\,d\Delta_1 + P_1\,d\Delta_1 + P_2\,d\Delta_2 + P_3\,d\Delta_3$$

if $(P_1 + dP_1)$, P_2 and P_3 were to be applied simultaneously.

$$S.E = \frac{1}{2}(P_1 + dP_1)(\Delta_1 + d\Delta_1)$$
$$+ \frac{1}{2}P_2(\Delta_2 + d\Delta_2)$$
$$+ \frac{1}{2}P_3(\Delta_3 + d\Delta_3)$$

so $\frac{1}{2}P_1\,d\Delta_1 + \frac{1}{2}P_2\,d\Delta_2 + \frac{1}{2}P_3\,d\Delta_3 = \frac{1}{2}dP_1\Delta_1$

$$\frac{1}{2}(P_1\,d\Delta_1 + P_2\,d\Delta_2 + P_3\,d\Delta_3) = \frac{1}{2}(dU - \frac{1}{2}dP_1\,d\Delta_1)$$

$$\frac{1}{2}(dU - \frac{1}{2}dP_1\,d\Delta_1) = \frac{1}{2}dP_1\Delta_1$$

$$\Rightarrow \frac{1}{2}dU = \frac{1}{2}dP_1\Delta_1$$

Neglecting small quantity of higher order.

$$\Rightarrow \frac{dU}{dP_1} = \Delta_1$$

if the moments are considered,

$$\frac{dU}{dM} = \theta$$

Finding Deflection using Castigliano's Method:-

- The S.E for the entire str. is differentiated w.r.t the load (P or M) to get the desired deflection.

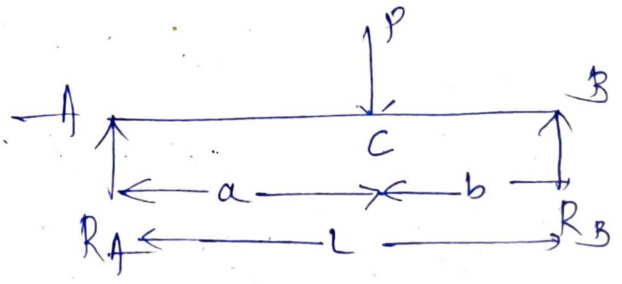
- If dummy load is used, then the load is zero.

$$\boxed{\frac{d}{dp}\int F(x,p)\,dx = \int \frac{d}{dp}\left[F(x,p)\right]dx}$$

Ex:- 3.11

A SSB of span L, carries a conc. load 'p' at a distance a from left hand side support. Using castigliano's thm determine the defln under the load. Assume uniform flexural rigidity.

Ans:- $R_A = \frac{Pb}{L}$



$R_B = \frac{Pa}{L}$

Portion	AC	CB
Origin	A	B
Limit	0 - a	0 - b
M	$\frac{Pb}{L}x$	$\frac{Pa}{L}x$
Flexural Rigidity	EI	EI

SE of the beam,

$$U = \int_0^a \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI}dx + \int_0^b \left(\frac{Pa}{L}x\right)^2 \times \frac{1}{2EI}dx$$

$$= \frac{P^2b^2}{L^2} \times \frac{1}{2EI}\int_0^a x^2 dx + \frac{P^2a^2}{L^2} \times \frac{1}{2EI}\int_0^b x^2 dx$$

$$= \frac{P^2b^2}{2EIL^2}\left[\frac{x^3}{3}\right]_0^a + \frac{P^2a^2}{2EIL^2}\left[\frac{x^3}{3}\right]_0^b$$

$$= \frac{P^2b^2}{6EIL^2}a^3 + \frac{P^2a^2}{6EIL^2}b^3$$

$$= \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

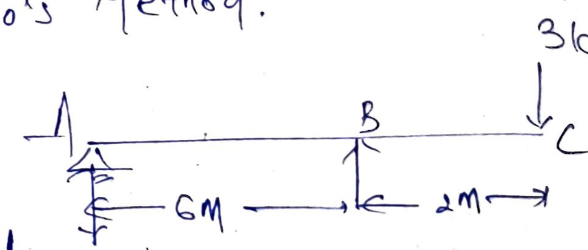
$$= \frac{P^2a^2b^2}{6EIL^2} \times L \qquad \because a+b = L$$

$$= \frac{P^2a^2b^2}{6EIL}$$

$$\Delta_C = \frac{\delta U}{\delta P} = \frac{Pa^2b^2}{3EIL}$$

Ex:- 3.12

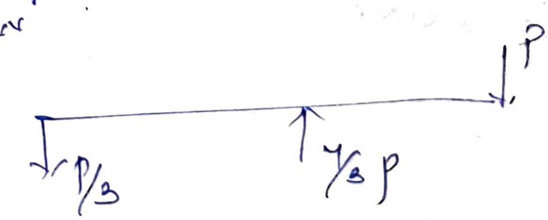
Determine the vertical defln at the free end and overhanging beam. Assume the rotation at A in the overhanging beam. Assume the rotation at A in the overhanging beam. Use castigliano's Method. Const. EI.



Ans:-

(1) Defln at C:-

taking 3 kN force as P, then



$R_B \times 6 = P \times 8$

$\Rightarrow R_B = \frac{4}{3}P \uparrow$

$\Rightarrow R_A = \frac{P}{3} \downarrow$

Reaction of 3 kN load is taken as!

Portion	AB	BC
Origin	A	C
Limit	0 – 6	0 – 2
M	$-\frac{P}{3}x$	$-Px$
Flexural Rigidity	EI	EI

$$U = \int \frac{M^2}{2EI}\,dx$$

$$= \int_0^6 \frac{P^2x^2}{9} \times \frac{1}{2EI}\,dx + \int_0^2 \frac{P^2x^2}{2EI}\,dx$$

$$= \frac{P^2}{18EI}\left[\frac{x^3}{3}\right]_0^6 + \left[\frac{P^2x^3}{6EI}\right]_0^2$$

$$= \frac{4P^2}{EI} + \frac{4}{3} \times \frac{P^2}{EI}$$

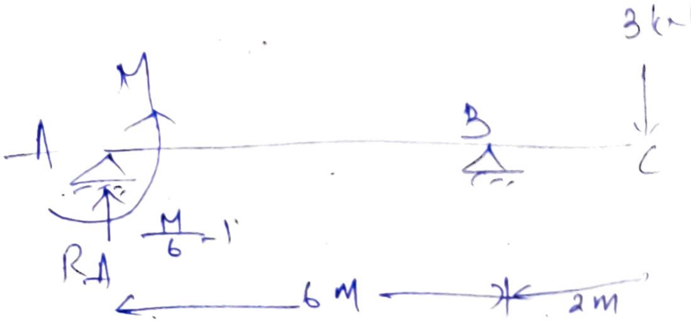
$$= \frac{5.333P^2}{EI}$$

$$\Delta_C = \frac{dU}{dP} = \frac{10.667P}{EI}$$

$P = 3$ kN

$$\Delta_C = \frac{32}{EI}$$

Rotation at A :- Apply a dummy moment M at A.



(Beam with dummy load at A)

$\Sigma M_B = 0$

$$R_A = \frac{M \cdot 6}{6}$$

$$= \frac{M}{6}$$

Portion	AB	BC
Origin	A	C
Limit	0 – 6	0 – 2
M	$(\frac{M}{6}-1)x - M$	$-3x$

$$U = \int_0^6 \left[(\tfrac{M}{6}-1)x - M\right]_0^2 \frac{1}{2EI}\,dx + \int_0^2 \frac{(-3x)^2}{2EI}\,dx$$

$$\frac{dU}{dM} = \int_0^6 2\left[(\tfrac{M}{6}-1)x - M\right]\left(\tfrac{x}{6}-1\right)\frac{dx}{2EI} + 0$$

Since, M → dummy moment = 0

$$\frac{dU}{dM} = \theta_A = \frac{1}{EI}\int_0^6 (-x)\left(\tfrac{x}{6}-1\right)dx$$

$$= \frac{1}{EI}\int_0^6 \left(-\frac{x^2}{6}+x\right)dx$$

$$= \frac{1}{EI}\left(-\frac{x^3}{18}+\frac{x^2}{2}\right)_0^6$$

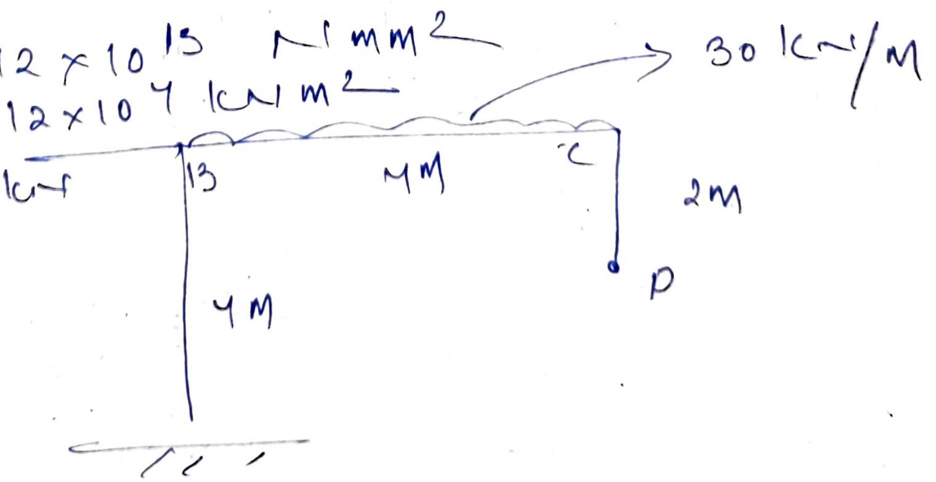
$$= \frac{6}{EI}$$

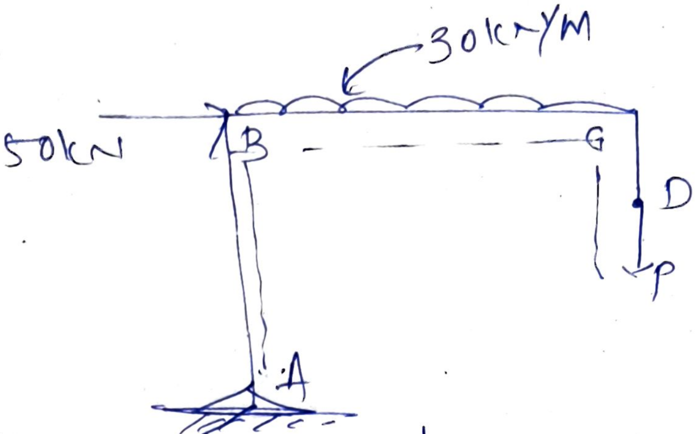
Ex:– 3.13

Determine the vertical & horizontal displacement at the free end 'D' in the frame.

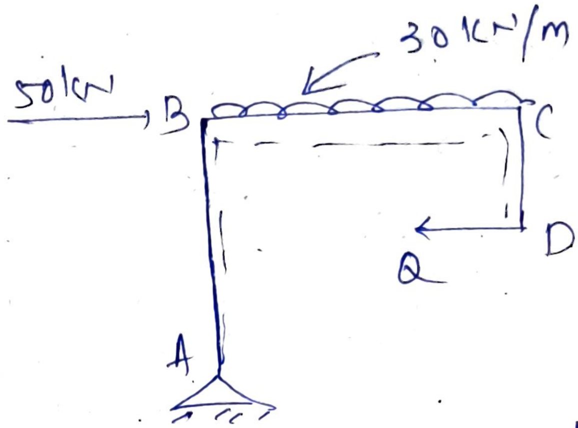
$EI = 12\times10^{13}$ N mm²
$= 12\times10^4$ kN m²

Ans:– 50kN





(Frame with dummy vertical load P at D)



Frame with dummy Horizontal load Q at D.

There is no load at D in vertical dirn, a dummy load P is applied at D in vertical dirn, in addition to given load.

— Moment → +ve → Tension

Portion	AB	BC	CD
Origin	B	C	D
Limit	0-4	0-4	0-2
M	$-(4P+240+50x)$	$-(Px+15x^2)$	0
FR	EI	EI	EI

There is no load in the horizontal dirn at D, a dummy load is applied.

Portion	AB	BC	CD
Origin	B	C	D
Limit	0-4	0-4	0-2
M	$-[Q(2-x)+240+50x]$	$-(2Q+15x^2)$	Qx
FR	EI	EI	EI

$$S.E,\ U = \int \frac{M^2}{2EI}\,dx$$

$$= \int_0^4 \frac{(4P+240+50x)^2}{2EI}\,dx + \int_0^4 \frac{(Px+15x^2)^2}{2EI}\,dx + 0$$

$$\Rightarrow \Delta = \frac{\delta U}{\delta P} = \int_0^4 2\,\frac{(4P+240+50x)}{2EI}\cdot 4\,dx + \int_0^4 2\,\frac{(Px+15x^2)\,x}{2EI}\,dx$$

P → dummy load, P = 0

$$\Delta_D = \int_0^4 \frac{4(240+50x)}{EI}\,dx + \int_0^4 \frac{15x^3}{EI}\,dx$$

$$= \frac{4}{EI}\left[240x+25x^2\right]_0^4 + \frac{15}{EI}\left(\frac{x^4}{4}\right)_0^4 = \frac{6400}{EI}$$

$$\Delta_{DV} = \frac{6400}{12\times10^4} = 0.533\ m = 53.33\ mm$$

$$U = \int_0^4 \frac{[Q(2-x)+240+50x]^2}{2EI}\,dx + \int_0^4 \frac{[(2Q+15x^2)]^2}{2EI}\,dx + \int_0^2 \frac{Q^2x^2}{2EI}\,dx$$

$$\Delta_{DH} = \frac{dU}{dQ}$$

$$= \int_0^4 \frac{2[Q(2-x)+240+50x](2-x)}{2EI}\,dx + \int_0^4 \frac{2[2Q+15x^2]\times 2}{2EI}\,dx + \int_0^2 \frac{2\cdot Q x^2}{2EI}\,dx$$

$Q = 0$

$$\Delta_{DH} = \int_0^4 \frac{(240+50x)(2-x)}{EI}\,dx + \int_0^4 \frac{30x^2}{EI}\,dx + 0$$

$$= \int_0^4 \frac{(480-140x-50x^2)}{EI}\,dx + \int_0^4 \frac{30x^2}{EI}\,dx$$

$$= \frac{1}{EI}\left[480x - 70x^2 - \frac{50x^3}{3}\right]_0^4 + \frac{1}{EI}\left[10x^3\right]_0^4$$

$$= \frac{373.33}{EI}$$

$$= \frac{373.33}{12\times 10^4}$$

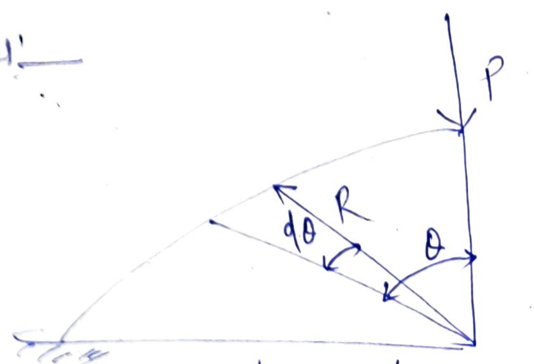
$= 0.0031$ m

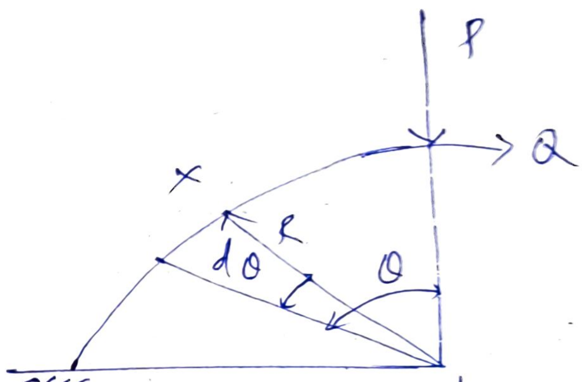
$= 3.1$ mm (→)

Ex!- 3.14

- A cantilevere beam is in the form of a quarter of a circle in the vertical plane and is s-t a vertical load 'p' at its free end. Find the vertical and horizontal displacements at the free end. Assume const. flexural rigidity.

- Ans!-





Vertical displacement of free end

$M = PR\sin\theta$

SE in the elemental length

$'Rd\theta' = \left(\frac{M^2}{2EI}\right) R\,d\theta$

$= \frac{P^2R^2\sin^2\theta}{2EI} R\,d\theta$

$= \frac{P^2R^3}{2EI} \times \frac{1-\cos 2\theta}{2} d\theta$

$U = \int_0^{\pi/2} \frac{P^2R^3}{2EI} \times \frac{1-\cos 2\theta}{2} d\theta$

$= \frac{P^2R^3}{4EI}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$= \pi P^2R^3/8EI$

$\Delta v = \frac{\delta U}{dP} = \frac{\pi P R^3}{4EI}$

Horizontal displacement

No Horizontal force at the free end apply a dummy horizontal force Q.

$S.E, U = \int_0^{\pi/2} \frac{[PR\sin\theta + QR(1-\cos\theta)]^2}{2EI} Rd\theta$

$M = PR\sin\theta + QR(1-\cos\theta)$

Horizontal displacement,

$\Delta_H = \frac{\delta U}{\delta Q}$

$= \int_0^{\pi/2} \frac{[PR\sin\theta + QR(1-\cos\theta)]}{EI} \times [R(1-\cos\theta)] Rd\theta$

$Q = 0$ then $\rightarrow$

$$\Delta H = \frac{1}{EI}\int_0^{\pi/2}\left(-\frac{PR\sin\theta}{EI}\right)\left[R(1-\cos\theta)\right]R\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}(\sin\theta - \sin\theta\cdot\cos\theta)\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}\left(\sin\theta - \frac{\sin 2\theta}{2}\right)d\theta$$

$$= \frac{PR^3}{EI}\left[\cos\theta - \frac{\cos 2\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{PR^3}{EI}\left(0 + \frac{1}{4} - 1 + \frac{1}{4}\right)$$

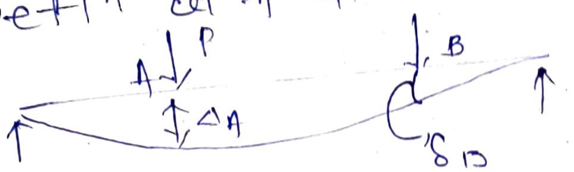
$$= \frac{-PR^3}{2EI}$$

$$\Rightarrow \Delta H = \frac{PR^3}{2EI} \longrightarrow \text{towards the support.}$$

Maxwell's thm of Reciprocal Deflection :-
↳ Clerk Maxwell (1864)

Displacement at pt. A due to the load at pt. B is same as displacement of pt. B due to the same load acting at pt. A, the displacements being measured in the dir^ns of the loads.

Defln at A due to load at P



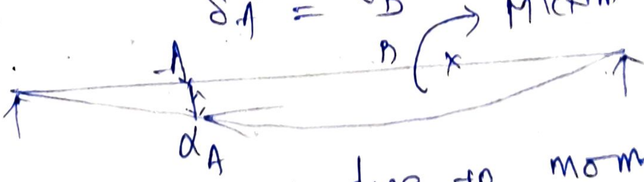
Defln at B due to load at A

the reciprocal thm is valid for linear as well as for rotational displacements.

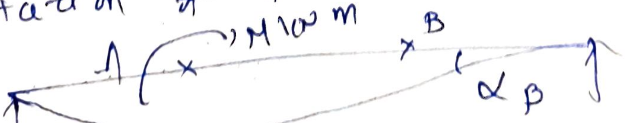
Ex:- δ_B is the displacement at B due to load P at A and δ_A is the displacement of A due to load P at B.

According to the thm,

$$\delta_A = \delta_B$$



Rotation of A due to moment at B



Rotation at B due to moment at A

Let α_A be the rotation at A due to the moment M at B and α_B be the rotation at B due to the moment M at A.

According to the thm,

$$\alpha_A = \alpha_B$$

Considering only linear displacement :—

When load P is acting at B, let the deflⁿ at A be δ_A and displacement at B be Δ_B.

$$\text{workdone} = \frac{1}{2} P \Delta_B$$

~~whe~~
when load P is acting at A, let the deflⁿ at A be Δ_A and deflⁿ at B be δ_B.

$$\text{workdone} = \frac{1}{2} P \Delta_A$$

Imagine the load P is applied 1st at B and then at A.

$$\text{External workdone} = \frac{1}{2} P \Delta_B + P \delta_B + \frac{1}{2} P \Delta_A$$

If load P is applied 1st at A and then at B,

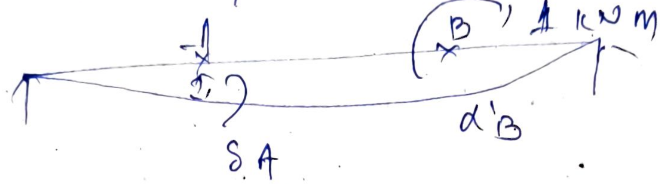
$$\text{workdone} = \frac{1}{2} P \Delta_A + P \delta_A + \frac{1}{2} P \Delta_B$$

Workdone when P is acting at both points A & B.

$$\frac{1}{2} P \Delta_B + P \delta_B + \frac{1}{2} P \Delta_A = \frac{1}{2} P \Delta_A + P \delta_A + \frac{1}{2} P \Delta_B$$

$$\Rightarrow \quad \delta_B = \delta_A \quad \text{(proved)}$$

* Maxwell thm in general is: Displacement at pt. A due to a unit load at pt. B is same as displacement at B due to the unit load at A, the displacements being measured in the dirctns of unit loads.



(Displacement due to unit moment at B)

Displacement due to unit load at A

$$\delta_A = \alpha_B$$

Proof,

Due to unit moment at B, the defln at A be δ_A and rotation at B be α_B'.

Due to moment 'M' at B defln at A is $M\delta_A$ and rotation at B is $M\alpha_B'$.

$$\text{Workdone} = \frac{1}{2} M \times M \times \alpha'_B$$

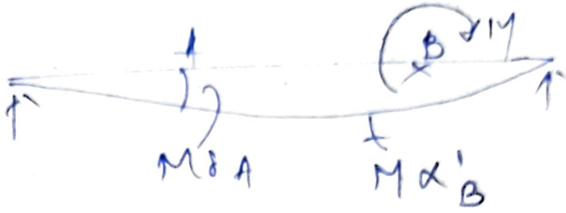
$$= \frac{1}{2} M^2 \alpha_B'$$

Due to unit load at A. Let the defln A be δ_A' and rotation at B be α_B'.

Hence, defln at A and rotation at B when load P is applied at A are $P\delta_A'$ and $P\alpha_B'$ respectively.

Workdone $= \frac{1}{2} P \times P \delta_A'$

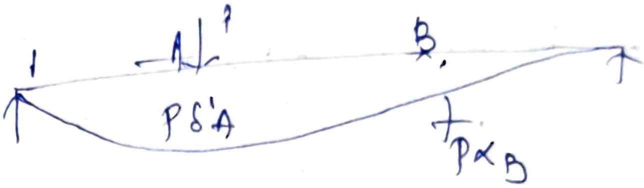
$= \frac{1}{2} P^2 \delta_A'$



consider in case when load M is applied before load P.

Due to M, workdone is $\frac{1}{2}M^2 \alpha_B'$ (Displacement due to moment M at B)

and the additional load P is applied, there will be an additional displacement



$P\delta_A'$ at A and $P\alpha_B$ at B. Displacement due to load P at A.

Total workdone $= \frac{1}{2} M^2 \alpha_B' + \frac{1}{2} P^2 \delta_A' + MP\alpha_B$

1st Applying P before M,

$W.D = \frac{1}{2} PP\delta_A' + PM\delta_A + \frac{1}{2} MM\alpha_B'$

$\frac{1}{2} M^2 \alpha_B' + \frac{1}{2} P^2 \delta_A' + MP\alpha_B$

$= \frac{1}{2} P^2 \delta_A + PM\delta_A + \frac{1}{2} M^2 \alpha_B'$

$\Rightarrow \alpha_B = \delta_A$ (proved)

Deflections of pin-jointed plane frames :—

Introduction :→

The trusses supporting sloping roofs and bridge decks are the examples of pin-jointed plane frames.

- These frames are usually made with steel sections.
- These slender steel members are connected at the ends by riveting or by welding to form a series of triangles.
- In this analysis, it is assumed that the members are pin-connected i.e the ends are perfect hinges and the loads act only at the joints.
- There is no BM & SF in the members.
- All members are s.t only direct forces. i.e tension or compression.

To find the defln of joints :—

1. Unit Load Method } To find the defln of a single joint at a time.
2. Castigliano's "
3. Angle weight " } To find the displacement of all joints at a time.
4. Joint displacement "
5. Williot-Mohr's " → Graphical Method → "
6. Stiffness Matrix Method → " & find the forces in the members. to finds its application in computer aided analysis.

* the displacements of joints are required mainly with two purposes. i.e.

1. to check whether the max^m displacement is with in the permissible limit.
2. to make use of it in the analysis of statically indeterminate str.

Unit load Method

By using energy eq^n.

the defl^n of a str., $\Delta = \int p'e\,dv$

where

Δ = Displacement at the pt. and in the direc^n of unit load applied.

p' = stress due to unit load

e = strain " " applied "

In case of pin-jointed frames, there is only one type of stress. i.e the direct stress.

- this stress may be different in different members but is const. at all pt.s in a member.

$$\boxed{\int p'\dot{e}\,dv = \Sigma p'eAL}$$

where,

Σ = To cover all members

A = cross-sectional area of members

L = length of the member

p' = stress due to unit load

$$\boxed{p' = \frac{k}{A}}$$

where,

k = force in the member due to unit load

e = strain due to given load

$$\left[e = \frac{P}{A} \times \frac{1}{E} = \frac{P}{AE} \right]$$

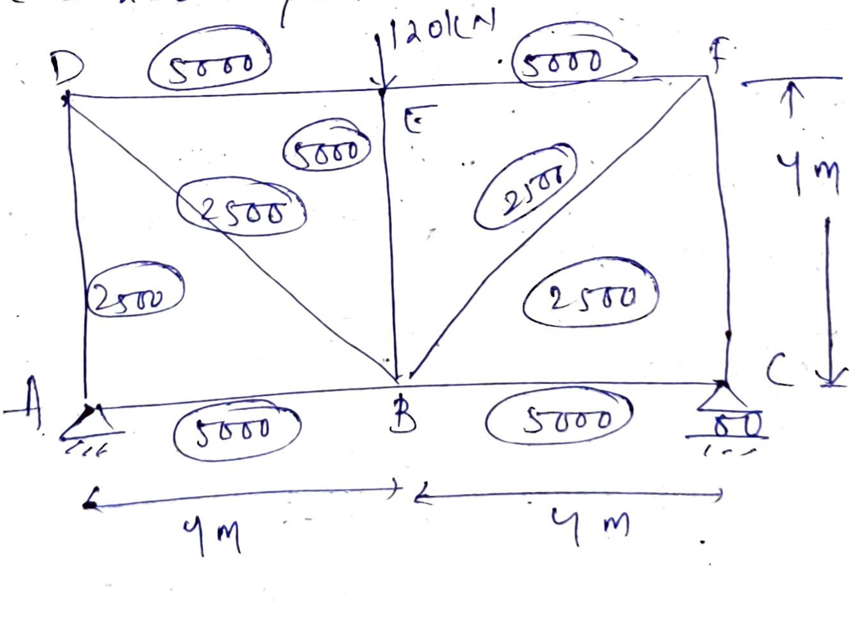
P = force in the member due to given loading.

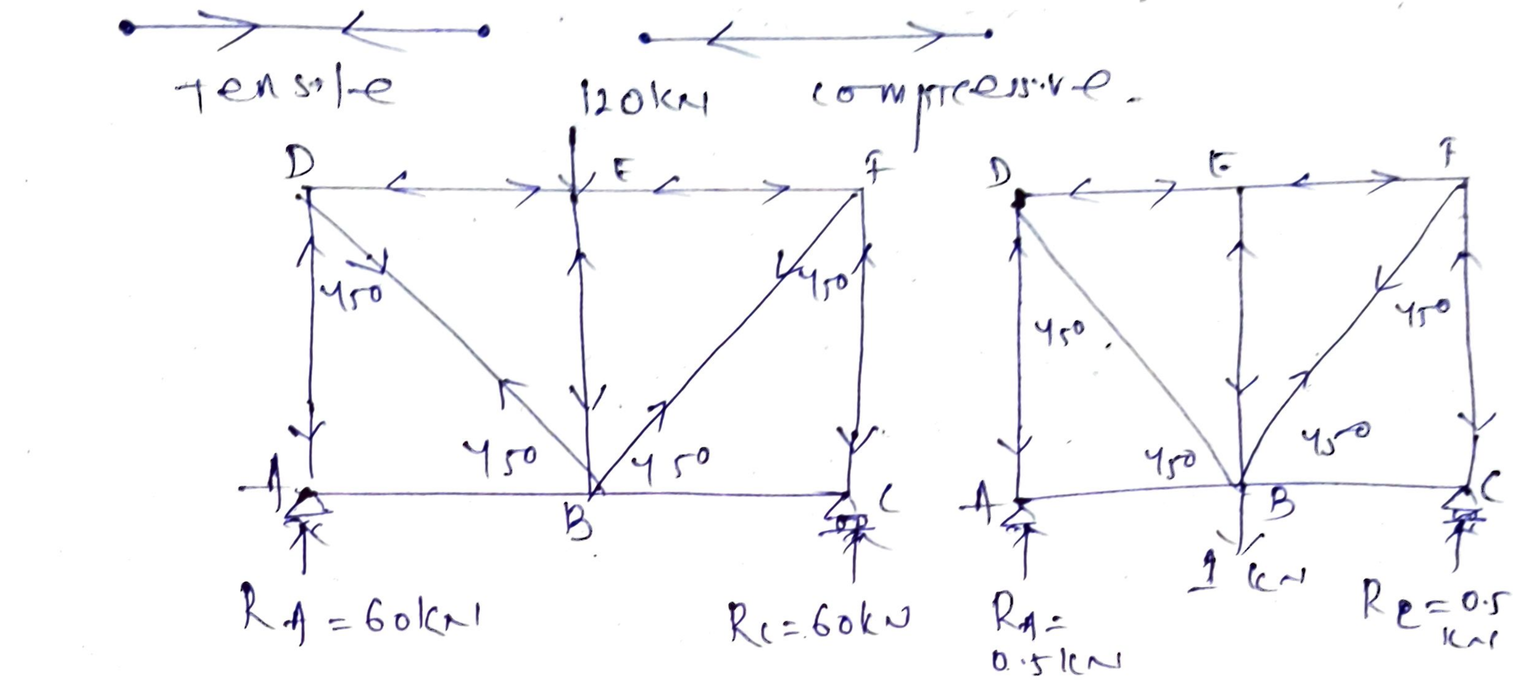
$$\Delta = \Sigma \frac{k}{A} \times \frac{P}{AE} \times AL = \Sigma \frac{PKL}{AE}$$

$$\Rightarrow \Delta = \Sigma \delta L \qquad \because \delta L = \frac{PL}{AE}$$ – extension/shortening of members

Ex:- 4.1

Find the vertical defl^n of the joint B in the truss loaded. The c/s area of the members in mm. Take, E = 200 kN/mm²





(Force due to given load)   (Force due to unit load at B)

P- forces due to given loading:—

Due to symmetry, $R_A = R_C = \frac{120}{2} = 60$ kN

From joint A, $P_{AD} = 60$ kN (comp.)

$R_{AB} = 0$

Joint D,

$P_{BD} \sin 45^\circ = 60^\circ \Rightarrow P_{BD} = 60\sqrt{2}$ kN (T)

$P_{DE} = P_{BD} \cdot \cos 45^\circ = 60$ kN (C)

At E, $P_{EB} = 120$ kN (C)

Due to unit load:—

$R_A = R_C = 0.5$ kN

Joint A, $F_{AD} = 0.5$ kN (C)

$F_{AB} = 0$

Joint D, $F_{BD} \times \frac{1}{\sqrt{3}} = 0.5 \Rightarrow F_{BD} = 0.5\sqrt{3}$ (T)

$F_{DE} = F_{BD} \times \frac{1}{\sqrt{2}} = 0.5$ kN (C)

Joint E, $F_{BE} = 0$

$$\Delta = \frac{\sum PkL}{A \cdot E}$$